UNIT - II

Likelihood Ratio Test (LRT) (or) Likelihood Ratio Criterian:

The method of maximum likelihood gives estimate which possess some optimum properties under certain conditions, a test procedure which is closely to related to this Likelihood ratio method was introdused, by Neyman Pearson for testing simple hypothesis or composite hypothesis.

Consider random variable X with probability density function $f(x, \theta)$.

The set Ω. which is the set of all possible value of θ is called the parameter space.

For example $X \sim N(\mu, \sigma^2)$ then the parameter space.

$$\Omega \text{ or } \Theta = \{(\mu, \sigma^2); -\infty \leq \mu < \infty; \sigma^2 > 0\} \text{ or.}$$

$$= \{(\mu, \sigma^2); -\infty \leq \mu < \infty; 0 < \sigma^2 < \infty\}$$

Consider a general family of distribution

$\{f(x_i, \theta_1, \theta_2 \dots \theta_k\}$ $\theta_i \in \Omega$ $i = 1, 2 \dots k\}$.

The null Hypothesis H_0 will state that the parameter belongs some sub spaces is 'C' of the parameter space Ω

Let $x_1, x_2, \ldots x_n$ random sample of size 'n' $(n>1)$ From a population with density function $f\{(x_i, \theta_1, \theta_2 \ldots \theta_k)\}$

Where Ω.

The parameter space in the totality of all parameters $\{(\theta_1, \theta_2 \ldots \theta_k)\}$ can be assumed to test the null hypothesis (H_0).

$$H_0 : (\theta_1, \theta_2 \ldots \theta_k) \in C$$

(Vs)

$$H_1 : (\theta_1, \theta_2 \ldots, \theta_k) \in \bar{C}$$

Where $\bar{C} = \Omega - C$

The likelihood function of the sample observation is given by $L(x_i, \theta) = \prod_{i=1}^{n} f(x_i, \theta_1, \theta_2 \ldots \theta_k)$

The criterian of the likelihood ratio test is defined as the Ratio of 2 maximum likelihood function is

$$\lambda = \frac{L_1}{L_0} > k$$

Where L_0 and L_1 are the maximum likelihood function with respect to the parameters.

The Quantity 'λ' is a function of sample observations only and does not involve any parameter, Thus λ is a random variable The critical region testing H_0 vs H_1 in an interval $0 < \lambda < \lambda_0$ Were λ_0 is some number (<1) determined by the distributions of λ and desired probability of type I error

λ_0 is given by the equation $\alpha = P(\lambda > \lambda_0 / H_0)$

Thus a test that has the critical region defined as $0 < \lambda < \lambda_0$ and $\alpha = P(\lambda > \lambda_0 | H_0)$ is LRT for testing H_0

Applications of LRT : (Likelihood Ratio Test)

- ⇒ Test for the mean of a normal population
- ⇒ Test for the equality of means of two normal population
- ⇒ Test for the equality of means of several normal population
- ⇒ Test for the variance of a normal population
- ⇒ Test for the equality of variances of Two normal population
- ⇒ Test for the equality of variances of several normal population

PROPERTIES OF LRT :

The LRT principle is an important testing one. If we are testing simple null hypothesis (H_0) Vs simple alternative hypothesis H_0. Then the LRT principle leads to the same test as given by NP Lemma, This suggestion that the likelihood ratio test has some desirable probability (desirable properties) especially large sample.

1) Under certain condition $-2 \log_e^{\lambda}$ has an assymptotic χ^2 distribution.

2) Under certain assumption LRT is consistent.

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TEST FOR THE MEAN OF NORMAL DISTRIBUTION :

Let $x_1, x_2, \ldots x_n$ be a random sample of size n from a normal population, with μ, σ^2 were μ & σ^2 are known may be unknown.

Suppose, we want to test the null hypothesis

H_0 such that $\mu = \mu_0$, $0 < \sigma^2 < \infty$ and

H_1 such that $\mu \neq \mu_0$, $0 < \sigma^2 < \infty$

In this case the parameter space Θ is given by

$\Theta \Rightarrow \{\mu, \sigma^2\}$ $-\infty < \mu < \infty$; $0 < \sigma^2 < \infty$ and sample space Θ_0

Determine by the null hypothesis is given by

$$\Theta_0 = \{(\mu, \sigma^2) ; \mu = \mu_0 ; 0 < \sigma^2 < \infty\}$$

We know that the likihood ratio criterian is given by

$$\lambda = \frac{L(\hat{\theta})}{L(\theta)} \Rightarrow \left(\frac{s^2}{\sigma_0^2}\right)^{n/2}$$

We know that under H_0 the test statistics

$$t = \frac{\bar{x} - \mu_0}{S/\sqrt{n}}$$

Were $S^2 = \frac{1}{n-1} \sum_{j=1}^{n} (x_i - \bar{x})^2$

$$= \frac{ns^2}{n-1}$$

Thus $t = \frac{\bar{x} - \mu_0}{s/\sqrt{n-1}} \sim t(n-1)$ .df.

$$\lambda = \frac{1}{\left(1 + t^2/n-1\right)} = \phi(t^2)$$

The likelihood ratio test for testing H_0. against H_1.

Consist in finding a critical region of the type $0 < \lambda < \lambda_0$

Were λ is given by $\int_0^{\lambda_0} g(\lambda | H_0)\, d\lambda = \alpha$. Which requires

The distribution of λ_0 under H_0 in this case, it's not

necessary to uptain t - distribution since $\lambda = \phi(t^2)$ is

a Monotonic function (< ∞ > series are called) of t^2 and

test can be carried out with t^2 as a likelihood

criterian as this λ, therefore the testing

$$\left.\begin{aligned} H_0 &= \mu = \mu_0 \\ H_1 &= \mu \neq \mu_0 \end{aligned}\right\} \sigma^2$$ unknown, We have two

tailed test defined as follows. df,

$$|t| = \frac{(\sqrt{n-1})(\bar{x} - \mu_0)}{d} \quad \text{or}$$

$$\left| \frac{(\sqrt{n-1})(\bar{x} - \mu_0)}{d} \right| \sim t_{(n-1)}(\alpha/2) \text{ do.}$$

$$= \frac{\bar{x} - \mu_0}{d/\sqrt{n-1}} \sim t_{(n-1)} \text{ df.}$$

C-V = t-V = "

C-V < t-V = accept H_0

$$\Rightarrow |t| < t_{(n-1)}\,\alpha/2$$

C-V > t-V = Reject H_0

Here H_0 may be accepted.

TEST FOR THE VARIENCE OF NORMAL POPULATION :

Let us consider, the problem of testing-2 the variance of normal population has a specified value σ_0^2 on the basis of a random sample $x_1, x_2 \dots x_n$ of size 'n' from a normal population with Mean (μ), Variance (σ^2) We want to test the null hypothesis

$H_0 : \sigma^2 = \sigma_0^2$, $H_1 = \sigma^2 \neq \sigma_0^2$. Here the parameter space $\Theta \Rightarrow \{(\mu, \sigma^2), -\alpha < \mu < \alpha, 0 < \sigma^2 < \alpha\}$

We know that, the likelihood ratio criterian is given by

$$\lambda = \frac{L(\hat{H}_0)}{L(\hat{H})} = \left\{\left(\frac{\alpha^2}{\sigma_0^2}\right)^{n/2}\right\}$$

We know that the λ is written as,

$$\lambda = \left\{\left(\frac{\alpha^2}{\sigma_0^2}\right)^{n/2} \cdot e^{-\frac{1}{2}\left(\frac{n\lambda^2}{\sigma_0^2} - n\right)}\right\}$$

We know that under H_0, the test statistic χ^2

$$\chi^2 = \frac{ns^2}{\sigma_0^2} \sim \chi^2 \text{ distribution. With } (n-1)$$

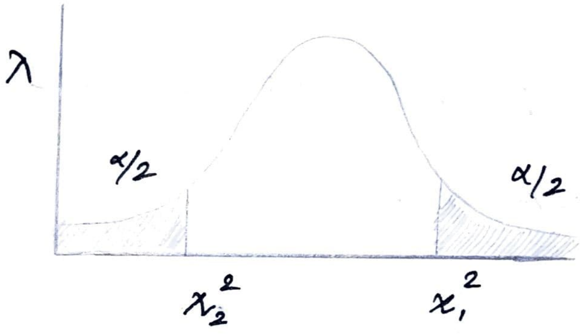
degrees of freedom is term of χ^2. We have

$$\lambda = \left(\frac{\chi^2}{n}\right)^{n/2} e^{-\frac{1}{2}\left(\frac{ns^2}{\sigma_0^2} - n\right)}$$

since λ is Monotonic function with χ^2

The test may be using x^2 as likelihood ratio criterian. Therefore the critical region of λ is given $0 < \lambda < \lambda_0$. since X^2 has x^2 distribution with $n-1$ degrees of freedom.

Therefore the critical region is determine by pair of interval $0 < x^2 < x_2^2$; $x_1^2 < x^2 < \alpha$



Where x_1^2 and x_2^2 determined such that the ordinates of area equal, Therefore critical region is shown drawing portion or shaded portion in the above diagram.

In other words x_1^2 and x^2 are defined by equations

$$P(x_0^2 > x_1^2) = \alpha/2 \quad \text{and critical region}$$

$$P(x^2 > x_2^2) = 1 - \alpha/2 \quad \text{accepting region.}$$

In otherwords,

$$x_1^2 = x^2 (n-1) \, \alpha/2 \ \text{d.f}$$

$$x_2^2 = x^2 (n-1) \, (1-\alpha/2) \ \text{df.}$$

Where,

$x^2 (n-1) \, \alpha/2$ is the Upper point of the x^2 distribution with $(n-1)$ d.f

Therefore the critical region for testing

$$H_0 : \sigma^2 = \sigma^2$$

$$H_1 : \sigma^2 \neq \sigma^2 \quad (\angle, \sigma, > \text{values})$$

Is a two tailed region or two tailed test is given by $\chi^2 (n-1)\ 1-\alpha/2 \cdot d.f$ and $\chi^2 (n-1)\ \alpha/2\ d.f$

Therefore in this case we have two tailed test

TEST FOR THE EQUALITY OF TWO VARIANCES OF TWO NORMAL POPULATION.

Consider two normal population $N \sim (\mu_1, \sigma_1^2)$ $N \sim (\mu_2, \sigma_2^2)$ when μ_1 and μ_2 and variances σ_1^2, σ_2^2 unspecified.

We want to test in our null hypothesis (H_0)

$$H_0 = \sigma_1^2 = \sigma_2^2 \quad Vs \quad H_1 = \sigma_1^2 \neq \sigma_2^2$$

If x_i, $(i = 1, 2 \dots n_1)$ and y_j, $(j = 1, 2 \dots n_2)$ (unspecified) be the independent random samples of sizes n_1 and n_2 from $N \sim (\mu_1, \sigma_1^2)$ and $N \sim (\mu_2, \sigma_2^2)$ respectively. Therefore the likelihood function

$$L = \left(\frac{1}{\sigma_1^2 2\pi}\right)^{n_1/2} \cdot e^{-\frac{1}{2}\left(\frac{x_i - \mu_1}{\sigma_1}\right)^2} \left(\frac{1}{2\pi\sigma_2^2}\right)^{n_2/2} \cdot e^{-\frac{1}{2}\left(\frac{x_i - \mu_2}{\sigma_2^2}\right)}$$

Therefore the parameter space,

$$\Theta = \left\{ (\mu_i, \sigma_i^2);\ i=1,2\ldots;\ \begin{matrix} -\infty < \mu_i < \infty \\ 0 < \sigma_i^2 < \infty \end{matrix} \right\}\ i=1,2\ldots$$

$$\lambda = \frac{L(\hat{\Theta_0})}{L(\hat{\Theta})}$$

$$F = s_1^2 / s_2^2 \sim (n_1-1)(n_2-1)\ df.$$

$$F = \frac{s_1^2/n_1}{s_2^2/n_2}$$

$$F = n_2 S_1^2 / n_1 S_2^2$$

Thus λ is a Monatonic of f distribution and hence,
The test can be carried f - distribution, Therefore the
critical region $0<\lambda<\lambda_0$ can we equal to be given by
pair of intervals $F \leq F_1$, $F \geq F_2$ were F_1 and F_2 determine
so that under H_0, $F \leq F_1 = \alpha/2$, $F \geq F_2 = 1-\alpha/2$ since under H_0 follows f - distribution
with (n_1, n_2) df

For testing $H_0 = \sigma_1^2 = \sigma_2^2$, $H_1 : \sigma_1^2 \neq \sigma_2^2$

We have two tailed test (F-test). Therefore the
critical region is given by $F > F_1 (n_1-1)(n_2-1)\ \alpha/2$ d.f.,

$F \leq F_2 (n_1-1)(n_2-1)\ 1-\alpha/2$ d.f.