Year	Subject Title	Sem	Sub Code
2018–19 Onwards	Core X: STATISTICAL INFERENCE – II	VI	18BST62C

Objective: To understand distributions by analyzing population data.

UNIT I

Testing of Hypothesis - Statistical Hypothesis - Simple and Composite Hypothesis, Null and Alternative Hypothesis - Two Types of Errors-Critical Region- Level of significance and Power of a Test - Most Powerful Test - Uniformly Most Powerful Tests -Neyman-Pearson Lemma.

UNIT II

Tests Based on N P lemma - Likelihood Ratio test – Definition - Test for Mean and Variance for normal population (One Sample Only).

UNIT III

Tests of Significance - Large Sample Tests - Mean, difference of Means, proportion-difference of proportions. Small Sample Tests - t-test for Mean, difference of Means, Paired t-test - Correlation Co-efficient

UNIT IV

F-test for variance ratio - Chi-Square Test – Contingency Tables -Yate's correction – Test for Goodness of Fit and Independence of Attributes.

UNIT V

Non-Parametric Tests: Advantages and limitations-Sign test, Run Test, Median Test and Mann-Whitney 'U' Test (One Sample and Two Sample Problems) - Kolmogorov's Smirnov One Sample Test- Kruskal Wallis Test – simple problems.

Text Books:

1. S.C. Gupta, and V.K.Kapoor - Fundamentals of Mathematical Statistics, Sultan Chand & Sons, New Delhi, 11th Revised Edition, June 2012.
2. Rohatgi V.K., - Statistical Inference, John Wiley and Sons, New York, 2013.

Reference Books:

1. Lehmann, E.L - Testing Statistical Hypothesis (2nd Edition, 1986) Springer New York.

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SUBJECT : **STATISTICAL INFERENCE - II**
CLASS : **III BS.c., (Statistics)**
UNIT : I

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Testing of hypothesis

Parameter & Statistic: Statistical measure computed from the population are called parameter. Calculation from the sample observations are called Statistic.
Example: Population mean μ, population variance σ^2 – Parameter.
Sample mean $\bar{x}$, Sample variance s^2 - Statistic.

Statistical Hypothesis: Making decisions about the population on the basis of Sample information such decisions are called statistical decisions, for eg, we may wish to decide on the basis of sample data whether a new drug is really effective in curing a disease, whether one educational procedure is better than another or whether a given coin is biased.

In attempting to reach decisions it is useful to make assumptions about the populations involved such observations Which he may or may not be true are called statistical hypothesis. They are generally statistics about the probability distribution of the population.

Explain the simple and composite hypothesis: If the statistical hypothesis Specifies the population completely then it is termed as simple statistical hypothesis otherwise it is called as composite Statistical hypothesis.

Eg. : In a bivariate normal distribution , With two means , two variables and one correlation coefficient if a hypothesis determines , only one , two , three or four parameters it is called composite hypothesis, but if it determines all the five parameters in Addison to the normality of the distribution It is called simple hypothesis .

Define Test of Hypothesis: A test of hypothesis procedure which specifies a set of rules for decision whether to accept or reject the hypothesis under the consideration.
That is the testing of hypothesis is a procedure that help us to as certain the likelihood of hypothesized population parameter being correct by making use of the sample statistic. Statistical test of hypothesis play an important role in the biological , the agricultural, the medical science & also in the industry.
The two types of hypothesis in a statistical test are:

Null Hypothesis : A statistical hypothesis which is set up and whose validity is tested for possible rejection the basis of sample observations is called null hypothesis . that is any statement or any assumption about the distribution on no difference basis. It is denoted as H_0.

Alternative Hypothesis : Any hypothesis which is complementary to the null hypothesis is called alternative hypothesis. It is denoted by H_1.

Example: We want to test the null hypothesis for an average plant height in a plot of a plants is say170 cms. Now these two hypotheses can be written as

$$H_0 : \mu = 170,\ H_1 : \mu \neq 170 \ .\ (\mu > 170)\text{ or }(\mu < 170)$$

One Tailed & Two Tailed Tests: When the rejection region consist of two regions each associated with probability α , we call it a two tailed test. on the other hand , when the rejection region consis of only one region(either on the right or left associated with probability α, we call it as one tailed test .

Example: Suppose we want to test the mean weight of a variety of wheat is 30 bushels per hectare.

One tailed Test: H_0 : the mean weight of a variety of wheat is 30 bushels per hectare, that is $\mu = 30$.

H_1 : The mean weight can be more than 30 bushels per hectare, that is $\mu > 30$. (Right Tailed Test)

H_1 : The mean weight can be less than 30 bushels per hectare, that is $\mu < 30$. (Left Tailed Test)

Two Tailed Test: H_0 : the mean weight of a variety of wheat is 30 bushels per hectare, that is $\mu = 30$.

H_1 : The mean weight is not 30 bushels per hectare, that is $\mu \neq 30$. (Two Tailed Test, that is either $\mu > 30$ Or $\mu < 30$.)

Define type-I and type- II Errors: When a hypothesis H_0 is tested against an alternative hypothesis H_1 there arise one of the two types of errors.

When a null hypothesis is rejected when it is true, it is known as type- I error.

If the null hypothesis is accepted when it is false , it is type- II error.

Type- I Error = Reject H_0 ,when H_0 is true

Type- II Error = Accept H_0 , when H_0 is false.

P (Type- I Error) = α

P (Type- II Error) = β,

Symbolically, $P (X \in W / H_0) = \alpha$, Where $X = (x_1, x_2, \dots x_n)$

This implies $\int_W L_0 dx = \alpha$, where L_0 is the likelihood function of the sample observations under H_0 & $\int dx$ represents the n fold integral $\int \dots \int d\,x_1, \int d\,x_2, \dots \int d\,x_n$

Against , $P (X \in \overline{W} / H_1) = \beta$ This implies $\int_{\overline{W}} L_1 dx = \beta$, where L_1 is the likelihood function of the sample observations under H_1.

$$\int_W L_1 dx + \int_{\overline{W}} L_1 dx = 1$$

$$\int_W L_1 dx = 1 - \int_{\overline{W}} L_1 dx$$

$$\int_W L_1 dx = 1 - \beta$$

Definition of Level of Significance: α, the probability of type- I error is known as the LOS of the test, it is also called the size of the critical region.

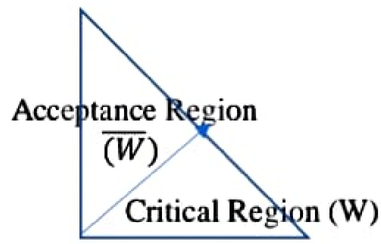
Explain the Power of the test: $1- \beta$ is called the power of the test and it gives the probability of making getting a correct decision.

If W is the critical region & $\overline{W}$ is the acceptance region, then the power of the test is derived as:

$$\begin{aligned} \beta &= P (X \in \overline{W} / H_1) \\ 1- \beta &= 1- P (X \in \overline{W} / H_1) \\ &= P (X \in W / H_1) \\ &= P \text{ (Rejecting } H_0 / H_1 \text{ is true)} \\ &= P \text{ (Rejecting } H_0 / H_0 \text{ is false)} \\ &= P \text{ (Correct Decision)} \end{aligned}$$

Explain the Critical Region: Let $x_1, x_2, \ldots x_n$ be the sample observations denoted by **O,** All the values of **O** will be aggregate of a sample and they constitute a space called sample space and is denoted by **S**.

The basis of the testing of hypothesis the division of the sample space into two exclusive regions W & S – W or $\overline{W}$ *, the null* hypothesis H_0 is rejected by the observations sample points fall in the W & if false in $\overline{W}$ we reject H_1, accept H_0, the region of rejecting H_0 when H_0 is true is that region of the outcome set. When H_0 is rejected if the sample points falls in that region and is called as the region of reject or Critical Region.



Explain Most Powerful Test: Let us consider the problem of testing a simple hypothesis

$H_0: \theta = \theta_0$ Vs a simple alternative hypothesis $H_1: \theta = \theta_1$.

Definition of Most Powerful Test: The critical region W is the Most Powerful critical Region of size α for testing $H_0: \theta = \theta_0$ Vs $H_1: \theta = \theta_1$. If

If i) $P(X \in W / H_0) = \int_W L_0 dx$

ii) $P(X \in W / H_1) \geq P(X \in W_1 / H_1)$ for every other Critical Region W_1 satisfying i)

Explain Uniformly most Powerful Test: Consider the testing of null hypothesis against the alternative hypothesis, $H_0: \theta = \theta_0$ Vs $H_1: \theta \neq \theta_0$.

In such a case , for predetermined α, the best test for H_0 is called UMP test of the level α.

Definition of Uniformly most Powerful Test (UMPT): The critical region W is called UMPT Critical Region of Size α testing $H_0: \theta = \theta_0$ Vs $H_1: \theta \neq \theta_0$.

If i) $P(X \in W / H_0) = \int_W L_0 dx$

ii) $P(X \in W / H_1) \geq P(X \in W_1 / H_1)$ for every $\theta \neq \theta_0$. What ever the region W_1 satisfies i) may be.

Problem:

1. Let x have a probability density function of the form

$$f(x,\theta) = \frac{1}{\theta} e^{-x/\theta}, \quad 0 < x < \infty, \quad \theta > 0$$

To test $H_0: \theta = 1$ vs $H_1: \theta = 2$ and the critical region $x \geq 0.5$. Find

i) size of TYPE-I Error
ii) size of TYPE-II Error
iii) power of the test

Solution:-

Given $H_0: \theta = 1$, $H_1: \theta = 2$, $CR = x \geq 0.5$

we know that

$$w = \{x \in w / H_0\} = CR$$

$$w = \{x : x \geq 0.5 / H_0\} - CR$$

$$\bar{w} = \{x : x \leq 0.5 / H_1\} - AR$$

$$\bar{w} = P(x \in \bar{w} / H_1) - AR$$
$$P(x < 0.5 / H_1)$$

i) $\alpha = P(x \in w / H_0)$

$$= P(x \geq 0.5 / H_0) / \theta = 1$$

$$= P(0.5 \leq x \leq \infty) / \theta = 1$$

$$\alpha = \int_w f(x)\, dx$$

$$\alpha = \int_{0.5}^{\infty} \frac{1}{\theta} e^{-x/\theta} / \theta = 1$$

$$= \frac{1}{\theta} \int_{0.5}^{\infty} e^{-x/\theta} . dx / \theta = 1$$

$$\int e^{-x/\theta} dx = \frac{e^{-x/\theta}}{-1/\theta} = -\theta e^{-x/\theta}$$

$$= \frac{1}{\cancel{\theta}} [-\cancel{\theta} e^{-x/\theta}]_{0.5}^{\infty} / \theta = 1$$

$$= -[e^{-x/\theta}]_{0.5}^{\infty} / \theta = 1$$

$$= -[e^{-x}]_{0.5}^{\infty}$$

$$= -[e^{-\infty} - e^{-0.5}] = -[0 - 0.6065]$$

TYPE I Error = 0.6065

(5)

ii) $\beta = P(x \in \bar{\omega} / H_1)$

$= P(x \le 0.5 / H_1)$

$= P(0 \le x \le 0.5)$

$= \int_0^{0.5} \frac{1}{\theta} e^{-x/\theta} dx \quad / \theta = 2$

$= \frac{1}{\theta} \int_0^{0.5} e^{-x/\theta} dx \quad / \theta = 2$

$= \frac{1}{\theta} \left[(-\theta \cdot e^{-x/\theta}) \right]_0^{0.5} \quad / \theta = 2$

$= -\left[e^{-x/2} \right]_0^{0.5}$

$= -\left[e^{-0.5/2} - e^{-0} \right]$

$= -\left[e^{-0.25} - e^{-0} \right]$

$= -[0.7788 - 1]$

$= -0.7788 + 1$

$\boxed{\beta = 0.2212}$

$\int e^{-x/\theta} dx = \frac{e^{-x/\theta}}{-1/\theta} = -\theta \cdot e^{-x/\theta}$

TYPE-II Error $\beta = 0.2212$

iii) $1 - \beta = P(x \in \omega / H_1)$

$= P(x \geq 0.5 / \theta = 2)$

$= P(0.5 \le x \le \infty) / \theta = 2$

$1 - \beta = \int_{0.5}^{\infty} \frac{1}{\theta} e^{-x/\theta} dx \quad / \theta = 2$

$= \frac{1}{\theta} \left[-\theta \cdot e^{-x/\theta} \right]_{0.5}^{\infty} \quad / \theta = 2$

$= -\left[e^{-x/2} \right]_{0.5}^{\infty}$

$= -\left[e^{-\infty/2} - e^{-0.5/2} \right]$

$= -\left[e^{-\infty} - e^{-0.25} \right] = -[0 - 0.7788]$

$\boxed{1 - \beta = 0.7788}$

2. Find the probability of Type-I Error, probability of Type-II Error and power of the test. while the testing $H_0 : \theta = 1$ vs $H_1 : \theta = 3$ when the probability density function is $f(x) = 1/\theta$, $0 < x < \theta$ and the CR is $0 \le x \le 1.5$ — X-f-10M

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Solution:-

Given $H_0 : \theta = 1$ v.s $H_1 : \theta = 3$

$f(x) = \frac{1}{\theta}$, $0 < x < \theta$

and the CR is $0 \le x \le 1.5$

$w : P(x \le 1.5 / H_0)$ - CR

$\bar{w} : P(x \ge 1.5 / H_1)$ - AR

$\therefore$ Type-I Error $\alpha = P(x \le 1.5 / H_0)$

$$= \int_0^{1.5} 1/\theta \cdot dx \;/ \theta = 1$$

$$= \frac{1}{\theta} \int_0^{1.5} dx \;/ \theta = 1$$

$$= \frac{1}{1} \cdot [x]_0^{1.5}$$

$$= [1.5 - 0]$$

Type-I Error $\boxed{\alpha = 1.5}$

Type-II Error $\beta = P(x \ge 1.5 / H_1) = P(1.5 \le x \le \theta)$

$$= \int_{1.5}^{\theta} \frac{1}{\theta} \, dx \;/ \theta = 3$$

$$= \frac{1}{\theta} \int_{1.5}^{\theta} dx \;/ \theta = 3$$

$$= \frac{1}{3} [x]_{1.5}^{\theta} \quad / \theta = 3$$

$$= \frac{1}{3} [\theta - 1.5] \;/ \theta = 3$$

$$= \frac{1}{3} [3 - 1.5]$$

$$= \frac{1}{3}^{(1.5)}$$

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$$= \frac{1.5}{3}$$

$$\boxed{\beta = 0.5}$$

power of a test

$$1-\beta = P(x \in w / H_1)$$

$$= P(x \le 1.5 / \theta = 3)$$

$$= P(0 \le x \le 1.5) / \theta = 3$$

$$= \int_0^{1.5} 1/\theta \, dx \quad / \theta = 3$$

$$= 1/\theta \int_0^{1.5} dx \quad / \theta = 3$$

$$= 1/\theta \, [x]_0^{1.5}$$

$$= \frac{1}{3}[1.5 - 0]$$

$$= \frac{1}{3}[1.5]$$

$$\boxed{1-\beta = 0.5}$$

$1-\beta = P(x \in$

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iven the frequency function $f(x,\theta) = \frac{1}{\theta}$, nd that you are testing the null hypo $H_0: \theta = 1$ vs $H_1: \theta = 2$ by means of a sing bserved the value of x, what would be ze of the TYPE-I Error and TYPE-II Er you choose the interval i) $0.5 < x$ ii) $1 < x$ ~~as the~~ critica ~~gion~~ Also obtain the power of the test

lution:-

$W: \{ x \in w / H_0 \}$

$$w: \{ x : x \geq 0.5 / H_0 \} - CR$$

$P(x > 0.5) - CR$

TYPE-I Error $\alpha = P(0.5 \le x \le \theta / \theta = 1)$

$$= \int_{0.5}^{1} 1/\theta \cdot dx \quad / \theta = 1$$

$= \frac{1}{\theta} \int_{0.5}^{1} dx \quad / \theta = 1$ 8

$= \frac{1}{\theta} [x]_{0.5}^{1} \quad / \theta = 1$

$= \frac{1}{1} [1 - 0.5]$

$= \frac{1}{1} [0.5]$

$\boxed{\alpha = 0.5}$

ii) TYPE - II Error $\beta = P(x \in \bar{\omega} / H_1)$

$\beta = P(x \in \bar{\omega} / H_1).$

$= P(x \leq 0.5 / H_1)$

$= \int_{0}^{0.5} \frac{1}{\theta} dx \quad / \theta = 2$

$= \frac{1}{\theta} [x]_{0}^{0.5} \quad / \theta = 2$

$= \frac{1}{2} [0.5 - 0]$

$= \frac{1}{2} [0.5]$

$\boxed{\beta = 0.25}$

iii) power of the test

$1 - \beta = 1 - 0.25$

$\boxed{1 - \beta = 0.75}$

ii) $\omega : \{x \in \omega / H_0\}$ - C.R

$\{1 \leq x \leq 1.5\}$ - C.R

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$\alpha = P\{1 \leq x \leq 1.5 / \theta = 1$

$= \int_{1}^{1.5} \frac{1}{\theta} dx \quad / \theta = 1$

$= \frac{1}{\theta} \int_{1}^{1.5} dx \quad / \theta = 1$

$= \frac{1}{\theta} [x]_{1}^{1.5} \quad / \theta = 1$

$= \frac{1}{1} [1.5 - 1]$

$= 1\,[0.5]$ (9)

$\boxed{\alpha = 0.5}$

$$\beta = 1 - \alpha$$

$$= 1 - \int_1^{1.5} \frac{1}{\theta}\, dx \;/ H_1 : \theta = 2$$

$$= 1 - \frac{1}{\theta}[x]_1^{1.5} \;/ \theta = 2$$

$$= 1 - \frac{1}{2}[1.5 - 1]$$

$$= 1 - \frac{1}{2}[0.5]$$

$$= 1 - \frac{0.5}{2}$$

$$= 1 - 0.25$$

$$\boxed{\beta = 0.75}$$

Hence the find ~~of~~ power of the test

$$1 - \beta = 1 - 0.75$$

$$= 0.25$$

4. If $x \geq 1$ is the critical region for testing $H_0 : \theta = 2$ vs $H_1 : \theta = 1$ on the basis of the single observation from the population $f(x, \theta) = \theta e^{-\theta x}$ $0 \leq x \leq \infty$ obtain the value of TYPE-I Error and TYPE-II Error

<u>Solution:-</u>

$$w = \{x : x \geq 1\} - CR$$

$$\bar{w} : \{x : x \leq 1\} - AR$$

$$\alpha = P(x \in w / H_0)$$

$$= P(1 \leq x \leq \infty) / \theta = 2$$

$$= \int_1^{\infty} \theta \cdot e^{-\theta x}\, dx \;/ \theta = 2$$

$= \theta \int_1^{\infty} e^{-\theta x}\, dx \;/ \theta=2$

$= \theta \left(\frac{-e^{-\theta x}}{\theta} \right)_1^{\infty} \;/ \theta=2$

$= \frac{\theta}{\theta} \left(-e^{-\theta x}\right)_1^{\infty} \;/ \theta=2$

$= \frac{2}{2} \left(-e^{-2x}\right)_1^{\infty}$

$\because e^{-\infty}=0$

$= -\left(e^{-2(\infty)} - e^{-2(1)}\right)$

$= -\left(e^{-\infty} - e^{-2}\right)$

$= -\left(0 - e^{-2}\right)$ (or) $e^{-2} = 0.1353$

$\boxed{\alpha = \frac{1}{e^2}}$

$\beta = P(x \in \bar{w} / H_1)$

$= P(x \leq 1 / \theta=1)$

$= P(0 \leq x \leq 1) / \theta=1$

$= \int_0^1 \theta e^{-\theta x} . dx \;/ \theta=1$

$= \theta \int_0^1 e^{-\theta x} dx \;/ \theta=1$

$= \theta \left(\frac{-e^{-\theta x}}{\theta} \right)_0^1 \;/ \theta=1$

$= \frac{1}{1} \left(-e^{-1x}\right)_0^1$

$\beta = (1 - 1/e)$

$= -\left(e^{-1(1)} - e^{-1(0)}\right)$

$1 - 1/e = e^{-1}$

$\beta = 1 - 1/e$

$= -\left[e^{-1} - e^{0}\right]$

$= -\left[e^{-1} - 1\right] \Rightarrow 1 - e^{-1}$

$\boxed{\beta = 1 - \frac{1}{e}}$

$1 - e^{-1}$

$= 1 - 0.3679$

$\beta = 0.6321$

Power of the test.

$1 - \beta = 1 - \left(1 - \frac{1}{e}\right)$

$= \cancel{1} - \cancel{1} + \frac{1}{e}$

$\boxed{1 - \beta = \frac{1}{e}}$ (or) $e^{-1} = 0.3678.$

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5. Let p be the probability that a coin will [fall] head d single toss in order to test H_0 : [$p = \frac{1}{2}$] vs $H_1 : p = \frac{3}{4}$ The coin is tossed 5 times and [H_0] is rejected if more than 3 heads are obtained. Find the probability of Type - I, type - II and p[ower]

Solution:-

$$H_0 : P = \frac{1}{2} \quad , \quad H_1 : P = \frac{3}{4}$$

If the random variable x denotes the number of heads in n tosses of a coin, then $x \sim B(n, p)$ that

$$P(X = x) = \binom{n}{x} p^x q^{n-x} = 5_{C_x} P^x q^{5-x}$$

the CR is given by $w = \{x : x \geq 4\}$

AR $\bar{w} = \{x : x \leq 3\}$

Probability of type - I error $\alpha = P(x \in w / H_0)$

$= P(x \geq 4 / P = \frac{1}{2})$

$\Rightarrow P(x = 4 / P = \frac{1}{2}) + P(x = 5 / P = \frac{1}{2})$

$= \binom{5}{4} (\frac{1}{2})^4 (\frac{1}{2})^{5-4} + \binom{5}{5} (\frac{1}{2})^5 (\frac{1}{2})^{5-5}$

$= 5 (\frac{1}{2})^5 + 1 (\frac{1}{2})^5$

$= 6 (\frac{1}{2})^5 = \frac{6}{32} = \frac{3}{16}$

$\alpha = \frac{3}{16}$

$5c_4 = \frac{5}{1}$

$5c_5 = 1$

Probability of Type - II Error $\beta = P(x \in \bar{w} / H_1)$

$= P(x \leq 3 / P = \frac{3}{4})$

$= 1 - P(x > 3 / P = \frac{3}{4})$

$= 1 - \{P(x = 4 / P = \frac{3}{4}) + P(x = 5 / P = \frac{3}{4})\}$

$= 1 - [\binom{5}{4} (\frac{3}{4})^4 (\frac{1}{4})^{5-4} + \binom{5}{5} (\frac{3}{4})^5 (\frac{1}{4})^{5-5}]$

$= 1 - [5 (\frac{3}{4})^4 (\frac{1}{4})^1 + 1 (\frac{3}{4})^5]$

$$= 1 - \left(\frac{3}{4}\right)^4 \left(\frac{5}{4} + \frac{3}{4}\right)$$

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$$= 1 - \frac{81}{256} \times \frac{8}{4}$$

$$= 1 - \frac{81}{128}$$

$$\beta = \frac{47}{128}$$

Hence to find power of the test

$$1 - \beta = 1 - \frac{47}{128}$$

$$= \frac{128 - 47}{128}$$

$$1 - \beta = \frac{81}{128}$$

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6. Find the value of α, β, power of the test when $f(x) = \theta \cdot e^{-\theta x}$, $x \geq 0$. The problem is to test $H_0 : \theta = 1$ vs $H_1 : \theta = 2$ when the CR is known to be $x \geq 1$

<u>Solution:-</u>

Given $H_0 : \theta = 1$ vs $H_1 : \theta = 2$

CR : $w = \{x : x \geq 1\}$; AR $= \bar{w} = \{x : x \leq 1\}$

$$\alpha = P(x \in w / H_0)$$

$$= P(x \geq 1 / \theta = 1)$$

$$= \int_1^\infty f(x, \theta)\, dx \,/\, \theta = 1$$

$$= \int_1^\infty \theta \cdot e^{-\theta x} dx \,/\, \theta = 1$$

$$= \int_1^\infty e^{-x} dx$$

$$= \left(\frac{e^{-x}}{-1}\right)_1^\infty$$

$$= (-e^{-x})_1^\infty$$

$$= (-e^{-\infty} + e^{-1}) = 0 + \frac{1}{e}$$

$$\alpha = \frac{1}{e} = e^{-1} = 0.3679$$

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$$\beta = P(x \notin w / H_1)$$

$$= P(x \le 1 / \theta = 2)$$

$$= \int_0^1 f(x, \theta)\, dx \,/\, \theta = 2$$

$$= \int_0^1 \theta \cdot e^{-\theta x}\, dx \,/\, \theta = 2$$

$$= 2 \int_0^1 e^{-2x}\, dx$$

$$= 2 \left(\frac{e^{-2x}}{-2} \right)_0^1$$

$$= \left(-e^{-2x} \right)_0^1$$

$$= -(e^{-2} + e^{0}$$

$$= 1 - e^{-2}$$

$$= 1 - \frac{1}{e^2} \qquad = 1 - e^{-2} = 1 - 0.1353$$

$$= 0.8647$$

$$\beta = \frac{e^2 - 1}{e^2}$$

Power of the test

$$1 - \beta = 1 - \left(\frac{e^2 - 1}{e^2} \right)$$

$$= \frac{\cancel{e^2} - \cancel{e^2} + 1}{e^2}$$

$$1 - \beta = \frac{1}{e^2} \qquad = 0.1353$$

State and prove
Neyman Pearson Lemma.

Statement:
Let $x_1, x_2, \ldots x_n$ be a random sample of size 'n' drawn from some population with pdf $f(x, \theta)$, where θ is an unknown parameter. Let $L(\theta)$ be the likelihood fn. of the sample $x_1, x_2 \ldots x_n$. Let k be a +ve no. then the best critical Region (BCR), w of size 'α' for testing the simple hypotheses $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$ is given by

$$w = \left\{ X \in S, \frac{f(X, \theta_1)}{f(X, \theta_0)} > k \right\}$$

$$\Rightarrow w = \left\{ X \in S, \frac{L_1}{L_0} > k \right\} \quad \text{--- (1)}$$

$$\bar{w} = \left\{ X \in S, \frac{L_1}{L_0} < k \right\} \quad \text{--- (2)}$$

where w is the critical Region and $\bar{w}$ is the acceptance region.

Proof.
We are given $P(X \in w / H_0) = \int_w L_0 \, dx = \alpha$ — (3)

The power of the test is

$$P(X \in w / H_1) = \int_w L_1 \, dx = 1 - \beta \quad (4)$$

In order to establish the lemma, we have to prove that there exists no other critical Region of size less than or equal to α.

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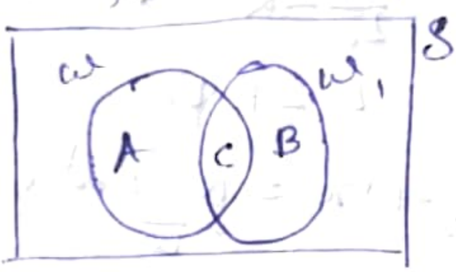
Which is More powerful than ω.

Let ω_1 be another CR of size $\alpha_1 \leq \alpha$, and power of the test $1-\beta_1$, So that we have

$$P(x \in \omega_1 / H_0) = \int_{\omega_1} L_0 \, dx = \alpha_1 \quad - (5)$$

$$P(x \in \omega_1 / H_1) = \int_{\omega_1} L_1 \, dx = 1-\beta_1 \quad - (6)$$

we have to prove that $1-\beta \geq 1-\beta_1$



Let $\omega = A \cup C$, $\omega_1 = B \cup C$.

C May be empty that is ω and ω_1 may be disjoint.

If $\alpha_1 \leq \alpha$, we have

$$\int_{\omega_1} L_0 \, dx \leq \int_{\omega} L_0 \, dx$$

$$\Rightarrow \int_{B \cup C} L_0 \, dx \leq \int_{A \cup C} L_0 \, dx$$

$$\Rightarrow \int_{B} L_0 \, dx \leq \int_{A} L_0 \, dx \qquad \text{(Since C is disjoint)}$$

$$\int_{A} L_0 \, dx \geq \int_{B} L_0 \, dx \quad - (7)$$

i) gives $\frac{L_1}{L_0} > k \ \forall \ x \in \omega$

$\Rightarrow L_1 > k\,L_0 \;\forall\; x \in w$

$\Rightarrow \int L_1 dx > k \cdot \int_w L_0 dx$

$\Rightarrow \int_A L_1 dx > k \int_A L_0 dx$

$\Rightarrow \int_A L_1 dx > k \int_A L_0 dx \geq k \cdot \int_B L_0 dx \rightarrow$ (7a)

from (7)

That is $k \int_B L_0 dx \leq k \int_A L_0 dx$ — (8)

Also (1) gives, $\frac{L_1}{L_0} < k \;;\; \forall\; x \in \bar{w}$

$\Rightarrow L_1 < k \cdot L_0$

$\Rightarrow \int_{\bar{w}} L_1 dx \leq k \int_{\bar{w}} L_0 dx$

Since $B \subset \bar{w}$, $\int_B L_1 dx \leq k \cdot \int_B L_0 dx \leq \int_A L_1 dx$ from (7a)

$\Rightarrow \int_B L_1 dx \leq \int_A L_1 dx$.

Adding $\int_C L_1 dx$ on both sides we get

$\Rightarrow \int_B L_1 dx + \int_C L_1 dx \leq \int_A L_1 dx + \int_C L_1 dx$

$\Rightarrow \int_{B \cup C} L_1 dx \leq \int_{A \cup C} L_1 dx \Rightarrow \int_{w_1} L_1 dx \leq \int_w L_1 dx$

$\Rightarrow 1 - \beta_1 \leq 1 - \beta$ from (6) and (4)

(or).

$1 - \beta \geq 1 - \beta_1$ Hence the proof.

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7. If $x_1, x_2, \dots x_n$ is a random sample from a distribution having the probability density function $f(x,\theta) = \begin{cases} \theta x^{\theta-1} , & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$ such that the best critical region for testing $H_0 : \theta = 1$ vs $H_1 : \theta = 2$ is $\{ C = \{x_1, x_2, \dots x_n\} \quad C \le \prod_{i=1}^{n} x_i$

Solution:-

Given $f(x,\theta) = \begin{cases} \theta x^{\theta-1} , & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

$$L = \prod_{i=1}^{n} f(x_i, \theta) = \prod_{i=1}^{n} \theta x^{\theta-1} =$$

$$L_0 = \prod_{i=1}^{n} \theta_0 x^{\theta_0 - 1} = \theta_0^n \prod_{i=1}^{n} x_i^{\theta_0 - 1}$$

$$L_1 = \prod_{i=1}^{n} \theta_1 x^{\theta_1 - 1} = \theta_1^n \prod_{i=1}^{n} x_i^{\theta_1 - 1}$$

By NP lemma, the CR $\omega = \{ x \in S , \frac{L_1}{L_0} > k \}$

$$\Rightarrow \frac{L_1}{L_0} > k \Rightarrow \frac{\theta_1^n \prod_{i=1}^{n} x_i^{\theta_1 - 1}}{\theta_0^n \prod_{i=1}^{n} x_i^{\theta_0 - 1}} > k$$

$\theta_0 = 1 , \theta_1 = 2$

$$\Rightarrow \frac{2^n \pi x_i^{2-1}}{1^n \pi x_i^{1-1}} > k$$

$$\Rightarrow \frac{2^n \pi x_i^1}{\pi x_i^0} > k$$

$$\Rightarrow 2^n \pi x_i^{1-0} > k$$

$$= \frac{\theta_1^n \pi x^{\theta_1 - 1}}{\theta_0^n \pi x^{\theta_0 - 1}}$$

$$= \left(\frac{\theta_1}{\theta_0}\right)^n \pi$$

$$= 2^n \pi$$

$$\Rightarrow 2^n \prod x_i > k$$

$$\prod x_i > \frac{1}{2^n} k$$

$$\prod_{i=1}^{n} x_i > c \quad \text{where } \frac{1}{2^n}\cdot k = c$$

8. Find the best critical Region for testing H_0 vs $H_1: \theta = \theta_1 (\leq \theta_0)$ given that $f(x,\theta) = \theta e^{-\theta x}$,

Solution:-

Given $f(x,\theta) = \theta e^{-\theta x}$

$$L = \prod f(x,\theta) = \prod \theta e^{-\theta x} = \theta^n$$

$$L_0 = \prod \theta_0 e^{-\theta_0 x} = \theta_0^n \prod e^{-\theta_0 x_i}$$

$$= \theta_0^n e^{-\theta_0 \Sigma x_i}$$

$$L_1 = \prod \theta_1 e^{-\theta_1 x} = \theta_1^n \prod e^{-\theta_1 x_i}$$

$$= \theta_1^n e^{-\theta_1 \Sigma x_i}$$

By NP lemma, the CR is $w = \{x \in S, \frac{L_1}{L_0} > k\}$

$$\therefore \frac{L_1}{L_0} > k \Rightarrow \frac{\theta_1^n e^{-\theta_1 \Sigma x_i}}{\theta_0^n e^{-\theta_0 \Sigma x_i}} > k$$

$$\Rightarrow \left(\frac{\theta_1}{\theta_0}\right)^n e^{-\Sigma x_i (\theta_1 - \theta_0)} > k$$

Taking log on both sides

$$n(\log \theta_1 - \log \theta_0) - \Sigma x_i (\theta_1 - \theta_0) > \log k$$

$$n(\log \theta_1 - \log \theta_0) + \Sigma x_i (\theta_0 - \theta_1) > \log k$$

$$\Sigma x_i (\theta_0 - \theta_1) = \log k - n(\log \theta_1 - \log \theta_0)$$

$$\Sigma x_i = \frac{\log k - n(\log \theta_1 - \log \theta_0)}{(\theta_0 - \theta_1)}$$

9. Examine whether a Best critical region exists for testing the null hypothesis $H_0: \theta = \theta_0$ vs $H_1: \theta > \theta_0$ for the parameter θ of the distribution $f(x,\theta) = \frac{1+\theta}{(x+\theta)^2}$

$1 \leq x < \infty$

Solution :-

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$$f(x,\theta) = \frac{1+\theta}{(x+\theta)^2}, \quad L = \prod f(x_i;\theta)$$

$$L = \prod \frac{(1+\theta)}{(x+\theta)^2} = (1+\theta)^n \prod_{i=1}^{n} \frac{1}{(x_i+\theta)^2}$$

$$L_0 = (1+\theta_0)^n \prod \frac{1}{(x_i+\theta_0)^2}$$

$$L_1 = (1+\theta_1)^n \prod \frac{1}{(x_i+\theta_1)^2}.$$

By NP lemma, the BCR is $w = \{x \in S, \frac{L_1}{L_0} > k\}$

$$\frac{L_1}{L_0} > k \Rightarrow \frac{(1+\theta_1)^n \prod \frac{1}{(x_i+\theta_1)^2}}{(1+\theta_0)^n \prod \frac{1}{(x_i+\theta_0)^2}} > k$$

$$\Rightarrow \frac{(1+\theta_1)^n \prod (x_i+\theta_0)^2}{(1+\theta_0)^n \prod (x_i+\theta_1)^2} > k$$

$$\Rightarrow \left(\frac{1+\theta_1}{1+\theta_0}\right)^n \prod \left(\frac{x_i+\theta_0}{x_i+\theta_1}\right)^2 > k$$

$$\prod \left(\frac{x_i+\theta_0}{x_i+\theta_1}\right)^2 > k\left(\frac{1+\theta_0}{1+\theta_1}\right)^n$$

Taking log on both sides,

$$\log\left(\prod\left(\frac{x_i+\theta_0}{x_i+\theta_1}\right)^2\right) > \log\left(k\left(\frac{1+\theta_0}{1+\theta_1}\right)^n\right)$$

$$2\Sigma \log\left(\frac{x_i+\theta_0}{x_i+\theta_1}\right) > \log k + n\log\left(\frac{1+\theta_0}{1+\theta_1}\right)$$

$2\Sigma \log\left(\frac{x_i+\theta_0}{x_i+\theta_1}\right) >$

Test criterion is $\Sigma \log\left(\frac{x_i+\theta_0}{x_i+\theta_1}\right)$. It is impossible to put this test criterion in the form a function of sample observation not depending on the hypothesis therefore there exists no test critical region in this cause.

(No test criterion)