20/2/21 UNIT - 5

NUMERICAL INTEGRATION

We know that $\int_a^b f(x)\,dx$ represents the area between $y=f(x)$, x-axis and the ordinates a and b.

The problem of numerical integration can be started as follows, Given a set of $(n+1)$ paired values, of the function $y=f(x)$ this not explicity its required to compute $\int_{x_0}^{x_n} y\,dx$ as we did in the case of interpolation or numerical differentiation we replace $f(x)$ by a polynomial $P_n(x)$ and obtain $\int_{x_0}^{x_n} P_n(x)\,dx$ which is approximately taken as the value of $\int_{x_0}^{x_n} f(x)\,dx$.

1. Trapezoidal Rule :

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\dots+y_{n-1})\right]$$

2. Simpson's 1/3rd Rule :

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+\dots)+4(y_1+y_3+\dots)\right]$$

3. Simpson's 3/8th rule :

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\dots y_{n-1}) + 2(y_3+y_6+y_9+\dots+y_n)\right]$$

Evaluate : $\int_{-3}^{3} x^4\,dx$ by using,

i) Trapezoidal's , Simpson's 1/3rd and 3/8th rule

also verify your results by actual integration

Here $y = f(x) = x^4$ the interval length is $b-a$. So we devide 6 equal intervals with $h = \frac{6}{6} = 1$

Now the table can be written as,

x :	-3	-2	-1	0	1	2	3
$f(x)$:	81	16	1	0	1	16	81

Trapezoidal Rule :

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\right]$$

$$\int_{-3}^{3} x^4\,dx = \frac{1}{2}\left[(81+81)+2(16+1+0+1+16)\right]$$

$$= \frac{1}{2}\left[(162)+68\right]$$

$$= \frac{230}{2} = 115$$

Simpson's $1/3^{rd}$ Rule :

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+\cdots)+4(y_1+y_3+y_5+\cdots)\right]$$

$$= \frac{1}{3}\left[(81+81)+2(1+1)+4(16+0+16)\right]$$

$$= \frac{1}{3}\left[162+4+128\right]$$

$$= \frac{294}{3}$$

$$= 98$$

Simpson's 3/8th rule:

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\Big[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\dots+y_{n-1})$$

$$+2(y_3+y_6+y_9+\dots+y_n)\Big]$$

$$= \frac{3}{8}\Big[(81+81)+3(16+1+1+16)+2(0)\Big]$$

$$= \frac{3}{8}\big[162+102\big]$$

$$= \frac{792}{8}$$

$$= 99$$

2) Evaluate Trapezoidal rule with

$$\int_0^1 \frac{1}{1+x^2}\,dx \qquad h = 0.2$$

x :	0	0.2	0.4	0.6	0.8	1
y :	1	0.962	0.862	0.735	0.610	0.5

Trapezoidal Rule:

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\Big[(y_0+y_n)+2(y_1+y_2+\dots+y_{n-1})\Big]$$

$$= \frac{0.2}{2}\Big[(1+0.5)+2(0.962+0.862+0.735+0.610)\Big]$$

$$= 0.1\Big[(1.5)+2(3.169)\Big]$$

$$= 0.1\big[1.5+6.338\big]$$

$$= 0.1\,(7.838)$$

$$= 0.7838$$

3) Evaluate $\int_{4}^{5.2} \log_e x$ using all three rules.

x:	4	4.2	4.4	4.6	4.8	5.0	5.2
$f(x)$:	1.38625	1.4350	1.4816	1.5260	1.5686	1.6094	1.6486

i) Trapezoidal Rule :

$$\int f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\dots+y_{n-1})\right]$$

$$= \frac{0.2}{2}\left[(1.38625+1.6486)+2(1.4350+1.4816+1.5260+1.5686+1.6094)\right]$$

$$= 0.1\left[3.03489 + 15.2412\right]$$

$$= 1.827609.$$

(ii) Simpson's 1/3rd rule :

$$= \frac{0.2}{3}\left[(1.38629+1.6486)+2(1.4816+1.5686)+4(1.4350+1.5260+1.6094)\right]$$

$$= 0.067\left[3.03489+6.1004+18.2816\right]$$

$$= 1.83693$$

(iii) Simpson's 3/8th rule :

$$= \frac{3(0.2)}{8}\left[(1.38629+1.6486)+3(1.4350+1.4816+1.5686+1.6094)+2(1.5260)\right]$$

$$= 0.075\left[3.03489+18.2838+3.052\right]$$

$$= 1.82780$$

4) Evaluate : $\int_0^6 \frac{1}{1+x}\,dx$

x :	0	1	2	3	4	5	6
y :	1	0.5	0.33	0.25	0.2	0.167	0.142

1. Trapezoidal Rule :

$$\int f(x)\,dx = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+\dots+y_5)\right]$$

$$= \frac{1}{2}\left[(1+0.142) + 2(0.5+0.33+\dots+0.167)\right]$$

$$= 0.5\left[1.142 + 5.9\right]$$

$$= 3.521$$

2. Simpson's 1/3rd rule :

$$\int f(x)\,dx = \frac{h}{3}\left[(y_0+y_6) + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right]$$

$$= \frac{1}{3}\left[(1.142) + 2(0.33+0.2) + 4(0.5+0.25+0.167)\right]$$

$$= 0.33\left[1.142 + 1.06 + 3.668\right]$$

$$= 1.9371$$

3. Simpson's 3/8th rule :

$$\int f(x)\,dx = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2(y_3)\right]$$

$$= \frac{3}{8}\left[(1.142) + 3(0.5+0.33+0.2+0.167) + 2(0.25)\right]$$

$$= 0.375\left[1.142 + 3.591 + 0.5\right]$$

$$= 1.962375$$

Note:

(i) In Trapezoidal rule, $y(x)$ is a linear function of x. This rule is the simplest one but it is least aqurate.

(ii) In Simpson's $1/3$rd rule, $y(x)$ is a polynomial of degree 2. To apply this rule n, the no. of intervals must be even. Ie the no. of ordinates must be odd.

(iii) In Simpson's $3/8^{th}$ Rule, $y(x)$ is a polynomial of degree 3. This rule is applicable if n, the no. of intervals is a multiple of 3.

Numerical Solution of Ordinary differential Equation:

In solving a differential equation for aproximate solution we find numerical values of $y_1, y_2, \ldots y_3$. corresponding to given numerical values of independent variable values $x_1, x_2, x_3 \ldots$ so that the order pairs $(x_1, y_1), (x_2, y_2) \ldots$ satisfy a particular solution through aprox. A solution of this type is called "point wise solution".

Suppose we required to solve

$\frac{dy}{dx} = f(x,y)$ with the initial condition

$y(x_0) = y_0$. by numerical solution of

differential equation.

Let $y(x_0) = y_0$, $y(x_1)$, $y(x_2)$... be the

exact solution.

There are three methods to solve

this equation.

1. Taylor series method
2. Euler method
3. Runge Kutta method.

i) Taylor Series method:

To find the numerical solution

of the equation $\frac{dy}{dx} = f(x,y)$ given the

initial condition $y(x_0) = y_0$.

The solution is,

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

1) Solve

$\frac{dy}{dx} = x+y$ given $y(1)=0$ and get

$y(1.1)$ and $y(1.2)$.

Solution:

Given: $x_0 = 1$, $y_0 = 0$, $h = 0.1$

$y_0' = x+y = 1+0 = 1$

$y_0'' = 1+y' = 1+1 = 2$

$y_0''' = 0+y'' = 0+2 = 2$

$y_0^{iv} = 0+y''' = 0+2 = 2$

$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \ldots$

$= 0 + \frac{0.1}{1}(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(2) + \frac{(0.1)^4}{24}(2)$

$= 0.1 + 0.01 + 0.00033 + 0.0000083$

$\boxed{y_1 = 0.11034}$

$x_1 = 1.1$, $y_1 = 0.11034$

$y_1' = x+y = 1.1 + 0.11034 = 1.2103$

$y_1'' = 1+y' = 1 + 1.2103 = 2.2103$

$y_1''' = 0+y'' = 0+2.2103 = 2.2103$

$y_1^{iv} = 0+y''' = 0+2.2103 = 2.2103$

$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{iv}$

$= 0.1103 + 0.1(1.2103) + \frac{(0.1)^2}{2}(2.2103) +$

$\frac{(0.1)^3}{6}(2.2103) + \frac{(0.1)^4}{24}(2.2103)$

$= 0.1103 + 0.12103 + 0.0110 + 0.00037 + 0.000009$

$= 0.24280$

$y_2 = 0.24280.$

2) Using Taylor series method, find the value of $y(0.1)$ and $y(0.2)$ given

$\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$, $h = 0.1$, $x_0 = 0$, $y_0 = 1$

Solution:

$y' = x^2 + y^2 = 0 + 1 = 1$

$y'' = 2x + 2y.y' = 2(0) + 2(1)(1) = 2$

$y''' = 2 + 2[y.y'' + y'.y'] = 2 + 2(1)(2) + 2(1)^2 = 8$

$y^{iv} = 2[y.y''' + y''.y'] + 2.2y'.y''$

$= 2y.y''' + 2y'y'' + 4y'.y''$

$= 2(1)(8) + 6(1)(2) = 16 + 12 = 28$

$y_1 = y_0 + \frac{h}{1!}y' + \frac{h^2}{2!}y'' + \frac{h^3}{3!}y''' + \frac{h^4}{4!}y^{iv} + \dots$

$= 1 + (0.1)(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(8) + \frac{(0.1)^4}{24}(28)$

$\boxed{y_1 = 1.111467}$

$x_1 = 0.1 \quad , y_1 = 1.11145$

$y_1' = x_1^2 + y_1^2 = (0.1)^2 + (1.11145)^2 = 0.01 + 1.235$

$= 1.245$

$y_1'' = 2x + 2y_1 y_1' = 2(0.1) + 2(1.11145)(1.245)$

$= 0.2 + 2.834 = 3.034$

$y_1''' = 2 + 2y.y'' + 2y'^2 = 2 + 2(1.11145)(3.034) + 2(1.275)^2$

$= 2 + 7.188 + 3.251 = 12.439$

$y_1^{iv} = 2y.y''' + 6y'y'' = 2(1.11145)(12.439) + 6(1.275)(3.234)$

$= 27.6506 + 24.7401 = 52.3907$

$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{iv}$

$= 1.11145 + 0.1(1.275) + \frac{(0.1)^2}{2}(3.234) + \frac{(0.1)^3}{6}(12.439)$

$+ \frac{(0.1)^4}{24}(52.3907)$

$\boxed{y_2 = 1.257411}$

3) Using Taylor's series method to compute $y(0.2)$ and $y(0.4)$ given, $\frac{dy}{dx} = 1 - 2xy$

$y(0) = 0$

Solution:

$x = 0, \; y_0 = 0 \quad , h = 0.2$

$y' = 1 - 2xy$

$y'' = 0 - 2(y + xy')$

$y''' = -2[y' + y' + xy'']$

$= -2[2y' + xy'']$

$y^{iv} = -2[y'' + y'' + y'' + xy''']$

$= -2[3y'' + xy''']$

4) Using Taylor Series method to find y at $x = (0.1)(0.2)$ ~~(0.4)~~ given $\frac{dy}{dx} = x^2 - y$

$y(0) = 1$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$

Solution:

$y' = x^2 - y = -1$

$y'' = 2x - y' = 1$

$y''' = 2 - y'' = 2 - 1 = 1$

$y^{iv} = -y''' = -1$

$y_1 = y_0 + \frac{h}{1!} y' + \frac{h^2}{2!} y'' + \frac{h^3}{3!} y''' + \frac{h^4}{4!} y^{iv}$

$= 1 + \frac{0.1}{1}(-1) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(1) + \frac{(0.1)^4}{24}(-1)$

$= 1 - 0.1 + 0.005 + 0.000167 - 0.00004$

$y_1 = 0.9051$

$x_1 = 0.1$, $y_1 = 0.9051$

$y' = x^2 - y = (0.1)^2 - 0.9051 = -0.8951$

$y'' = 2x - y' = 2(0.1) + 0.8951 = 1.0951$

$y''' = 2 - y'' = 2 - 1.0951 = 0.9049$

$y^{iv} = -y''' = -0.9049$

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \cdots$$

$$= 0.9051 + \frac{0.1}{1}(-0.8951) + \frac{(0.1)^2}{2}(0.6951) + \frac{(0.1)^3}{6}(2.6951) + \frac{(0.1)^4}{24}(2.6951)$$

$$= 0.9051 - 0.08951 - 0.00347 + 0.000449 - 0.0000112$$

$$= 0.81255$$

(ii) Euler method :

To solve $\frac{dy}{dx} = f(x,y)$ with initial tangent corresponding to $x = x$ in the initial x_0, x_1 the curve is approximated by the tangent therefore the value of y on the curve is approximated equal to the value of y on the tanget at (x_0, y_0) coresponding to $x = x_1$, $y = y_0 * f(x_0, y_0)$

Hence all the algorithm is,

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}, \quad n = 0, 1, \ldots$$

1) Given $y' = -y$, $y(0) = 1$ find the value of y at $x = 0.01 (0.01) (0.04)$ by euler method.

Solution:

$f(x,y) = -y$, $x_0 = 0$, $x_1 = 0.01$, $x_2 = 0.02, \ldots$

$y_0 = 1$, $h = 0.01$

$y_{n+1} = y_n + h f(x_n, y_n)$

$n=0$ $y_1 = y_0 + 0.01 f(x_0, y_0) = 1 + 0.01(-1) = 1 - 0.01 = 0.99$

$n=1 \Rightarrow y_2 = y_1 + 0.01 f(x_1, y_1)$

$= 0.99 + 0.01(-0.99) = 0.9801$

$n=2 \Rightarrow y_3 = y_2 + 0.01 f(x_2, y_2)$

$= 0.9801 + 0.01(0.9801) = 0.9703$

$n=3 \Rightarrow y_4 = y_3 + 0.01 f(x_3, y_3)$

$= 0.9703 + 0.01(-0.9703)$

$= 0.9606$

2) Using euler method, solve numerically

$y' = x+y, \; y(0) = 1$, for $x = 0.0(0.2)(1.0)$

solution:

$f(x,y) = x+y$, $x_0 = 0, x_1 = 0.2, x_2 = 0.4,$

$x_3 = 0.6, \; x_4 = 0.8, \; x_5 = 1.0$; $h = 0.2$

$n=0 \Rightarrow y_1 = y_0 + 0.2 f(x_0 y_0)$ $\qquad f(x_0, y_0) = x+y$

$= 1 + 0.2(1) = 1.2$ $\qquad = 0+1$

$= 1$

$n=1 \Rightarrow y_2 = y_1 + 0.2 f(x_1, y_1)$

$= 1.2 + 0.2(1.4) = 1.48$

$n=2 \Rightarrow y_3 = y_2 + 0.2 f(x_2, y_2)$

$= 1.48 + 0.2(1.88) = 1.856$

$n=3 \Rightarrow y_4 = y_3 + 0.2 f(x_3, y_3)$

$= 1.856 + 0.2(2.456)$

$= 2.232$

3) Solve numerically, $y' = y + e^x$

$y(0) = 0$ for $x = 0.2$ & 0.4, $h = 0.2$

given,

$x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$

$y_{n+1} = y_n + hf(x_n y_n)$

$n=0$, $y_1 = y_0 + 0.2 f(x_0 y_0)$

$= 0 + 0.2(1) = 0.2$

$n=1$, $y_2 = y_1 + 0.2 f(x_1, y_1)$

$= 0.2 + 0.2(1.42) = 0.484$

$n=2$, $y_3 = y_2 + 0.2 f(x_2, y_2)$

$= 0.484 + 0.2(1.97) = 0.878$

Runge Kutta Algorithm (2nd order)

To solve $\frac{dy}{dx} = f(x, y)$ given

$y(x_0) = y_0$.

second order Runge kutta algorithm is

$k_1 = hf(x, y)$

$k_2 = hf\left(x + \frac{h}{2}, y + \frac{k_1}{2}\right)$

and $\Delta y = k_2$, where $h = \Delta x$.

① Using R.K method 2nd order obtain the value of y at $x = 0.1, 0.2$; $y' = -y$

<u>Solution:</u>

$f(x, y) = -y$

$x_0 = 0,\ x_1 = 0.1,\ x_2 = 0.2,\ y_0 = 1$

$k_1 = hf(x, y) = 0.1(-1) = -0.1$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= 0.1 f\left(0 + \frac{0.1}{2}, 1 - \frac{0.1}{2}\right) = 0.1 f(0.05, 0.95)$

$= 0.1(-0.95) = -0.095$

$\Delta y = k_2 = -0.095$

$y_1 = y_0 + \Delta y = 1 - 0.095 = 0.905$

$x_1 = 0.1,\ y_1 = 0.95$

$k_1 = hf(x_1, y_1) = 0.1(-0.905) = -0.0905$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1 f\left(0.1 + \frac{0.1}{2}, 0.905 - \frac{0.9}{2}\right)$

$= 0.1 f(0.15, 0.86) = -0.086 = \Delta y$

$y_2 = y_1 + \Delta y$

$= 0.905 - 0.086 = 0.819$

$x_2 = 0.2,\ y_2 = 0.819$

2) Apply R.K method to find $y(0.2)$ given

$y' = x+y$, $y(0)=1$, $h=0.1$

<u>Solution:</u>

$f(x,y) = x+y$

$x_0=0$, $y_0=1$

$k_1 = hf(x,y) = 0.1\,(0,1) = 0.1(1) = 0.1$

$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right) = 0.1 f\left(0+\frac{0.1}{2}, 1+\frac{0.1}{2}\right)$

$= 0.1 f(0.05, 1.05)$

$= 0.1(1.20) = 0.120 = \Delta y$

$y_1 = y_0 + \Delta y = 1+0.120 = 1.120$

$x_1 = 0.1$, $y_1 = 1.120$

$k_1 = hf(x_1,y_1) = 0.1\,(0.1, 1.120)$

$= 0.1(1.220) = 0.1220$

$k_2 = hf\left(x_1+\frac{h}{2}, y_1+\frac{k_1}{2}\right)$

$= 0.1 f\left(0.1+\frac{0.1}{2}, 1.120+\frac{0.1220}{2}\right)$

$= 0.1(0.15+1.181) = 0.1(1.331)$

$= 0.1331 = \Delta y$

$y_2 = y_1 + \Delta y = 1.120 + 0.1331 = 1.2531$

$x_2 = 0.2$ $y_2 = 1.2531$