Unit - 4

Numerical differentiation

Using interpolation formula we can find the function $y = f(x)$ passing through $(n+1)$ ordered pairs (x_i, y_i) where $i = 0, 1, 2, \dots n$.

Numerical differentiation is a method of finding a derivative value of such curves at given $X = X_K$ whose $x_0 < X_K < X_n$

To get the derivative we first find the curve $y = f(x)$ through the points and then differentiate and get its values at the require point.

Derivative of Newton's Forward difference formula.

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{24}\Delta^4 y_0 + \dots\right] \quad (1)$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{(6u^2-18u+11)}{12}\Delta^4 y_0\right] \quad (2)$$

When $x = x_0 \Rightarrow u = 0$ and so ① & ②

becomes

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \dots\right] \rightarrow ③$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \dots\right] \rightarrow ④$$

Derivative of Newton's Backward Formula:

$$\left.\frac{dy}{dx}\right|_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{2v+1}{2}\nabla^2 y_n + \frac{3v^2+6v+2}{6}\nabla^3 y_n + \frac{4v^3+18v^2+22v+6}{24}\nabla^4 y_n + \dots\right] \rightarrow ⑤$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + (v+1)\nabla^3 y_n + \left(\frac{6v^2+18v+11}{12}\right)\Delta^4 y_n\right] \rightarrow ⑥$$

When $x = x_n \Rightarrow v = 0$ then ⑤ and ⑥

becomes,

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{\Delta^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \dots\right] \rightarrow ⑦$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \dots\right] \rightarrow ⑧$$

Derivatives using stirling's formula

$$\frac{dy}{dx} = \frac{1}{h}\left[\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) + u\Delta^2 y_{-1} + \frac{(3u^2-1)}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{12}(2u^3 - u)\Delta^4 y_{-2} + \cdots\right] \rightarrow (9)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_{-1} + \frac{u}{2}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{(6u^2-1)}{12}\Delta^4 y_{-2}\right] \rightarrow (10)$$

Sub $u=0$ in (9) and (10)

$$\frac{dy}{dx} = \frac{1}{h}\left[\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{60}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) \cdots\right] \rightarrow (11)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \cdots\right] \rightarrow (12)$$

Find the first 2 derivative of $x^{\frac{1}{3}}$ at $x = 50$ and $x = 56$ given the data below.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
50	3.6840			
		0.0244		
51	3.7084		-0.0003	
		0.0241		0
52	3.7325		-0.0003	
		0.0238		0
53	3.7563		-0.0003	
		0.0235		0
54	3.7798		-0.0003	
		0.0232		0
55	3.8030		-0.0003	
		0.0229		
56	3.8259			

We require $f'(x)$ at $x = 50$

So we use Newton's forward formula.

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \left.\frac{dy}{dx}\right|_{u=0}$$

$$= \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 \dots\right]$$

$$= \frac{1}{1}\left[0.0244 - \frac{1}{2}(-0.0003)\right] \because \Delta$$

$$= [0.0244 + 0.00015]$$

$$= 0.02455$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=50} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \dots\right]$$

$$= \frac{1}{1^2}[-0.0003]$$

$$= -0.0003$$

To get $f'(x) = 56$ we use Newton's backward formula.

$$\left.\frac{dy}{dx}\right|_{x=56} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 \ldots\right]$$

$$= \frac{1}{1}\left[0.0244 + \frac{1}{2}(-0.0003)\right]$$

$$= 0.0244$$

$$= \frac{1}{1}\left[0.0229 - 0.00015\right]$$

$$= 0.02275$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=56} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \ldots\right]$$

$$= \frac{1}{1}\left[-0.0003 + 0\right]$$

$$= -0.0003$$

2) Find first 2 derivatives of $\sqrt{x}$ for the values 15 and 25

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
15	3.873					
		0.250				
17	4.123		-0.014			
		0.236		0.002		
19	4.359		-0.012		-0.001	
		0.224		0.001		0.00
21	4.583		-0.011		0.001	
		0.213		0.002		
23	4.796		-0.009			
		0.204				
25	5.000					

To get $f'(x) = 15$ we use Newton's forward formula.

$$\left.\frac{dy}{dx}\right|_{x=15} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0\right]$$

$$= \frac{1}{2}\left[0.250 - \frac{1}{2}(-0.014) + \frac{1}{3}(0.002) - \frac{1}{4}(-0.001) + \frac{1}{5}(0.002)\right]$$

$$= \frac{1}{2}\left[0.250 + 0.007 + 0.00067 + 0.00025 + 0.0004\right]$$

$$= \frac{0.25832}{2} = 0.12916$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=15} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \Delta\right]$$

$$= \frac{1}{4}\left[-0.014 - (0.002) - \frac{(11)0.001}{12}\right]$$

$$= \frac{1}{4}\left[-0.014 - 0.002 - 0.0009167\right]$$

$$= -0.004229$$

To get $f'(x) = 25$ we use Newton's Backward formula.

$$\left.\frac{dy}{dx}\right|_{x=25} = \frac{1}{2}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n\right]$$

$$= \frac{1}{2}\left[0.204 + \frac{(-0.009)}{2} + \frac{(0.002)}{3} + \frac{(0.001)}{4} + \frac{(0.002)}{5}\right]$$

ans: $= 0.10041$

$$= \frac{1}{2}\left[0.204 - 0.0045 + 0.00067 + 0.00025 + 0.0004\right]$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=25} = \frac{1}{4}\Big[\nabla^2 y_n + \nabla^3 y_n + \nabla^4 y_n$$

$$= \frac{1}{4}\Big[-0.009 + 0.002 + 0.001$$

$$0.002\Big]$$

$$= \frac{1}{4} - 0.004$$

$$= 0.246$$

19/2/21 Find the value of $f'(0.5)$ using Stirling's Formula

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0.35	-3 1.521					
		-0.015				
			-0.003			
0.40	-2 1.506			0		
		-0.018			0.001	
0.45	-1 1.488		-0.003			-0.0[illegible]
				+.001		
		-0.021			-0.002	
$\boxed{0.50}$	0 1.467		-0.002			
		-0.023		-0.001		0.0[illegible]
0.55	1 1.444		-0.003		0.001	
		-0.026		0		
0.60	2 1.418		-0.003			
		-0.029				
0.65	3 1.389					

$$\frac{dy}{dx} = \cancel{\frac{1}{h}\Big[\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) + u\,\Delta^2 y_{-1}} + 3u^2.$$

$$= \frac{1}{h}\Big[\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2})$$

$$+ \frac{1}{60}(\Delta^5 y_{-2} + \Delta^5 y_{-3})$$

$$= \frac{1}{h}\Big[\frac{1}{2}(-0.023 - 0.021) - \frac{1}{12}(-0.001 + 0.001)$$

$$+ \frac{1}{60}(0.003 - 0.003)$$

$$= \frac{1}{0.05}(-0.022 + 0 + 0)$$

$$= -0.44$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \cdots\right]$$

$$= \frac{1}{0.05}\left[-0.002 - \frac{1}{12}(-0.002)\right]$$

$$= \frac{1}{0.05}\left[-0.002 + 0.000167\right]$$

$$= -0.0367$$

2) $x = 0.6$

x :	0.4	0.5	0.6	0.7	0.8
y :	1.5836	1.7974	2.0442	2.3275	2.6511

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.4	1.5836				
		~~0.2248~~ 0.2138			
0.5	1.7974		0.0330		
		0.2468		0.0035	0.003
0.6	2.0442		0.0365		
		0.2833		0.0038	
0.7	2.3275		0.0403		
		0.3236			
0.8	2.6511				

$$\frac{dx}{dy} = \frac{1}{h}\left[\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{60}(\Delta^5 y_{-2} + \Delta^5 y_{-3})\right]$$

$$= \frac{1}{0.1}\left[\frac{1}{2}(0.2833 + 0.2468) - \frac{1}{12}(0.0038 + 0.0035)\right]$$

$$= 10\left[0.26505 - 0.0006083\right] \cancel{+\frac{1}{60}}$$

$$= 2.644417$$