UNIT - II.

(7)

CENTRAL DIFFERENCE INTERPOLATION FORMULAE

$Ey_1 = y_2$

(1) GAUSS FORWARD FORMULA:

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots \quad \text{— (1)}$$

Let

$$\Delta^2 y_0 = \Delta^2 E y_{-1} = \Delta^2(1+\Delta)y_{-1}$$

$$= \Delta^2 y_{-1} + \Delta^3 y_{-1}$$

Sily

$$\left.\begin{array}{l}\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1} \\ \Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1}\end{array}\right\} \text{— (2)}$$

Sub (2) in (1).

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}[\Delta^2 y_{-1} + \Delta^3 y_{-1}] + \frac{u(u-1)(u-2)}{3!}[\Delta^3 y_{-1} + \Delta^4 y_{-1}] + \dots$$

$$= y_0 + u\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1} + \left[\binom{u}{2} + \binom{u}{3}\right]\Delta^3 y_{-1} + \left[\binom{u}{3} + \binom{u}{4}\right]\Delta^4 y_{-1} + \dots$$

$$= y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1}$$

(r)

$$+ \binom{u+1}{4}\Delta^4 y_{-1} + \binom{u+1}{5}\Delta^5 y_{-1} + \dots \quad \text{--- (3)}$$

As in (2)

$$\left.\begin{aligned} \Delta^4 y_{-1} &= \Delta^4 y_{-2} + \Delta^5 y_{-2} \\ \Delta^5 y_{-1} &= \Delta^5 y_{-2} + \Delta^6 y_{-2} \end{aligned}\right\} \text{--- (4)}$$

Subs. (4) in (3)

$$y(x) = y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1} +$$

$$\binom{u+1}{4}\left[\Delta^4 y_{-2} + \Delta^5 y_{-2}\right] +$$

$$\binom{u+1}{5}\left[\Delta^5 y_{-2} + \Delta^6 y_{-2}\right] + \dots$$

$$= y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_0 + \binom{u+1}{3}\Delta^3 y_{-1} +$$

$$\binom{u+1}{4}\Delta^4 y_{-2} + \left[\binom{u+1}{4}\binom{u+1}{5}\right]\Delta^5 y_{-2} + \dots$$

$$y(x) = y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{0_{-1}} + \binom{u+1}{3}\Delta^3 y_{-1} +$$

$$\binom{u+1}{4}\Delta^4 y_{-2} + \binom{u+2}{5}\Delta^5 y_{-2} + \dots$$

(2) GAUSS BACKWARD FORMULA:

Consider newton's forward interpolation formula.

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

But $\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$

$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$

$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$

$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$

$$y(x) = y_0 + u[\Delta y_{-1} + \Delta^2 y_{-1}] + \binom{u}{2}[\Delta^2 y_{-1} + \Delta^3 y_{-1}]$$

$$+ \binom{u}{3}[\Delta^3 y_{-1} + \Delta^4 y_{-1}] + \binom{u}{4}[\Delta^4 y_{-1} + \Delta^5 y_{-1}] + \cdots$$

$$= y_0 + u\Delta y_{-1} + \left[\binom{u}{1}\binom{u}{2}\right]\Delta^2 y_{-1} + \left[\binom{u}{2}\binom{u}{3}\right]\Delta^3 y_{-1}$$

$$+ \left[\binom{u}{3}\binom{u}{4}\right]\Delta^4 y_{-1} + \cdots$$

$$= y_0 + \binom{u}{1}\Delta y_{-1} + \binom{u+1}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1}$$

$$+ \binom{u+1}{4}\Delta^4 y_{-1} + \cdots$$

$$= y_0 + \binom{u}{1}\Delta y_{-1} + \binom{u+1}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\left[\Delta^3 y_{-2} + \Delta^4 y_{-2}\right.$$

$$\left. + \binom{u+1}{4}\left[\Delta^4 y_{-2} + \Delta^5 y_{-2}\right] + \cdots\right.$$

$$= y_0 + \binom{u}{1}\Delta y_{-1} + \binom{u+1}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-2}$$

$$+ \binom{u+2}{4}\Delta^4 y_{-2} + \cdots$$

QH/1 Gauss Forward Central difference formula.

Fd. $f(32)$

x:	25	30	35	40
y:	0.2707	0.3027	0.3386	0.3794

x	u	y	Δy	$\Delta^2 y$	$\Delta^3 y$
25	-1	0.2707			
			0.032		
30	0	$\boxed{0.3027}$		$\boxed{0.0039}$	
			$\boxed{0.0359}$		$\boxed{0.001}$
35	1	0.3386		0.0049	
			0.0408		
40	2	0.3794			

$$u = \frac{32 - 30}{5} = 0.4$$

$$y(x) = y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1}$$

$$= 0.3027 + \binom{0.4}{1}(0.0359) + \binom{0.4}{2}(0.0039) + \binom{1.4}{3}(0.001)$$

$$= 0.3027 + 0.4(0.0359) + \frac{(0.4)(0.4-1)}{2}(0.0039) + \frac{(1.4)(1.4-1)(1.4-2)}{6}(0.001)$$

$$= \underline{\underline{0.316}}$$

(3) Using Gauss backward find population for year 1936.

Year :	1901	1911	1921	1931	1941	1951
Pop :	12	15	20	27	39	52

x	$u = \frac{x - x_0}{h}$	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
1901	-4	12					
			3				
1911	-3	15		2			
			5		0		
1921	-2	20		2		3	
			7		3		-10
1931	-1	27		5		-7	
			$\boxed{12}$		$\boxed{-4}$		
1941	0	$\boxed{39}$		$\boxed{1}$			
			13				
1951	1	52					

$$y(x) = y_0 + \binom{u}{1}\Delta y_{-1} + \binom{u+1}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-2} + \binom{u+2}{4}\Delta^4 y_{-2} + \cdots$$

$$u = \frac{1936 - 1941}{10} = -5/10 = -0.5$$

$$= 39 + (0.5)(12) + \frac{(0.5)(0.5-1)(1)}{2} + \frac{(0.5)(0.5-1)(0.5-2)}{6}(-4)$$

$$= 39 \mp 6 \mp 0.125 - 0.25$$

$$= \cancel{44.625}\ 32.625$$

(4) Fd. $51°42'$ using Gauss backward

x: 50° 51° 52° 53° 54°

y: 0.6428 0.6293 0.6157 0.6018 0.5878

x	u	y	Δ	Δ^2	Δ^3	Δ^4
50	-2	0.6428				
			-0.0135			
51	-1	0.6293		-0.0001		
			$\boxed{-0.0136}$		$\boxed{-0.0002}$	$\boxed{0.0004}$
52	0	$\boxed{0.6157}$		$\boxed{-0.0003}$		
			-0.0139		0.0002	
53	1	0.6018		-0.0001		
			-0.014			
54	2	0.5878				

51 42
52

$$u = \frac{x-52^\circ}{1^\circ} = \frac{51^\circ 42' - 52^\circ}{1^\circ} = \frac{-18}{60'} = -0.3$$

$$y(x) = y_0 + \binom{u}{1}\Delta y_{-1} + \binom{u+1}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1} +$$

$$\binom{u+2}{4}\Delta^4 y_{-2} + \cdots$$

$$= 0.6157 + (-0.3)(-0.0136) + \frac{(0.7)(0.7-1)}{2}(-0.0003) +$$

$$\frac{0.7(0.7-1)(0.7-2)}{6}(-0.0002) + \frac{(1.7)(1.7-1)(1.7-2)(1.7-3)}{24}(0.0004)$$

$$= 0.6157 + 0.000408 + 0.0000315$$

$$+ 0.0000091 - 0.000025415$$

$$\simeq \underline{0.61979}$$

⑤ Forward → Fd y from $x = 0.68$.

x		y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
0.5	-3	0.1915						
			0.0173					
0.55	-2	0.2088		-0.0003				
			0.017		-0.0003			
0.60	-1	0.2258				0.0003		
				-0.0006				
			0.0164		0		-0.0001	
0.65	0	$\boxed{0.2422}$				$\boxed{0.0002}$		-0.0006
			$\boxed{0.0158}$	-0.0006	$\boxed{0.0002}$		$\boxed{-0.0007}$	
0.7	1	0.258		$\boxed{-0.0004}$		-0.0005		
			0.0154		-0.0003			
0.75	2	0.2734		-0.0007				
0.8	3	0.2881	0.0147					

$$u = \frac{x - x_0}{h} = \frac{0.68 - 0.65}{0.5} = \frac{0.03}{0.05} = 0.6$$

$$y(x) = y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_0 + \binom{u+1}{3}\Delta^3 y_{-1} + \binom{u+1}{4}\Delta^4 y_{-2} + \binom{u+2}{5}\Delta^5 y_{-2} +$$

$$= 0.2422 + (0.6)(0.0158) + \frac{(0.6)(-0.4)}{2}(-0.0004) + \frac{(1.6)(0.6)(-0.4)}{6}(0.0002)$$

$$+ \frac{(1.6)(0.6)(-0.4)(-1.4)}{24}(0.0002) + \frac{(2.6)(1.6)(0.6)(-0.4)(-1.4)}{120}(-0.0007)$$

$= 0.2422 + 0.00948 + 0.000048 - 0.0000128$

$+ 0.00000448 - 0.00000815$

$= 0.251711$

08/01

③ STIRLING'S FORMULA :

Consider Gauss forward formula,

$$y(x) = y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1} + \binom{u+1}{4}\Delta^4 y_{-2} + \binom{u+2}{5}\Delta^5 y_{-2} + \dots \quad (1)$$

Consider Gauss Backward Formula,

$$y(x) = y_0 + \binom{u}{1}\Delta y_{-1} + \binom{u+1}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-2} + \binom{u+1}{4}\Delta^4 y_{-2} + \dots \quad (2)$$

Adding ~~from~~ ① & ② we get,

$$2y(x) = 2y_0 + \binom{u}{1}[\Delta y_0 + \Delta y_{-1}] + \left[\binom{u}{2} + \binom{u+1}{2}\right]\Delta^2 y_{-1} + \binom{u+1}{3}[\Delta^3 y_{-1} + \Delta^3 y_{-2}] + \left[\binom{u+2}{4} + \binom{u+1}{4}\right]\Delta^4 y_{-2} + \dots$$

$$= 2y_0 + \binom{u}{1}[\Delta y_0 + \Delta y_{-1}] + \frac{2u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}[\Delta^3 y_{-1} + \Delta^3 y_{-2}] + \frac{2u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \cdots$$

$$\therefore y(x) = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

(4) BESSEL'S FORMULA:

Consider Gauss forward formula,

$$y(x) = y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1} + \binom{u+1}{4}\Delta^4 y_{-2} + \cdots \quad (1)$$

We know,

$$\left.\begin{aligned} \Delta y_0 &= y_1 - y_0 \\ y_0 &= y_1 - \Delta y_0 \\ y_{-1} &= y_1 - \Delta y_{-1} \\ \Delta^2 y_{-1} &= \Delta^2 y_1 - \Delta^3 y_{-1} \\ \Delta^4 y_{-1} &= \Delta^4 y_{-1} - \Delta^5 y_{-1} \end{aligned}\right\} \quad (2)$$

$\therefore$ ① becomes,

$$y(x) = \left(\frac{y_0}{2} + \frac{y_0}{2}\right) + (u\Delta y_0) + \left(\frac{1}{2}\frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{1}{2}\frac{u(u-1)}{2!}\Delta^2 y_{-1}\right)$$

$$+ \left(\frac{(u+1)(u)(u-1)}{3!}\Delta^3 y_{-1}\right) + \ldots \quad \text{——— ③}$$

Substitute ② in ③,

$$y(x) = \cancel{\frac{y_0}{2}} + \frac{y_0}{2} + \frac{1}{2}(y_1 - \Delta y_0) + u\Delta y_0 + \frac{1}{2}\frac{u(u-1)}{2!}\Delta^2 y_{-1}$$

$$+ \frac{1}{2}\frac{u(u-1)}{2!}\left[\Delta^2 y_0 - \Delta^3 y_{-1}\right] + \frac{(u+1)(u)(u-1)}{3!}\Delta^3 y_{-1} + \ldots$$

On Simplifying,

$$y(x) = \frac{y_0 + y}{2} + (u - 1/2)\Delta y_0 + \frac{u(u-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right]$$

$$+ \frac{(u - 1/2)(u)(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)(u)(u-1)(u-2)}{4!}\left[\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right] + \ldots$$

How to use center interpolation formula:

i) If the interpolation is required near the begin of the tabular values, use Newton Gregory Forward Interpolation Formula (Gauss forward)

ii) If the interpolation is required near the end of the tabular values, use Newton Gregory Backward Interpolation Formula. (Gauss Backward)

iii) If interpolation is required near the middle value of the table use either Stirling's or Bessel's Formula.

★ If $-1/4 \leq u \leq 1/4 \longrightarrow$ Stirling's Formula.
-0.25 0.25

★ If $1/4 \leq u \; 3/4 \longrightarrow$ Bessel's Formula.
0.25 0.75

Eg.

0.40

09/01 Fd. $e^{0.644}$ using i) Stirling's + ii) Bessel's

x		y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
0.61	-3	1.840431						
			0.018497					
0.62	-2	1.858928		0.000185				
			0.018682		0.000004			
0.63	-1	1.877610		0.000189		-0.000004		
			$\boxed{0.018871}$		$\boxed{0}$		+0.000006	
0.64	0	1.896481		$\boxed{0.000189}$		$\boxed{0.000002}$		-0.000007
			$\boxed{0.01906}$		$\boxed{0.000002}$		-0.000001	
0.65	1	1.915541		0.000191		0.000001		
			0.019251		0.000003			
0.66	2	1.934792		0.000194				
			0.019445					
0.67	3	1.954237						

$$u = \frac{x - x_0}{h} = \frac{0.644 - 0.64}{0.01} = 0.4$$

i) Stirling's Formula:

$$y(x) = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \dots$$

$$= 1.896481 + (0.4)\left[\frac{0.01906 + 0.018871}{2}\right] + \frac{(0.4)^2}{2}(0.000189)$$

$$+ \frac{(0.4)(0.4^2 - 1)}{6}\left[\frac{0.000002 + 0}{2}\right] + \frac{0.4^2(0.4^2-1)}{16}(0.000002)$$

$= 1.896481 + 0.007586 + 0.00001512 + 0.000000056$

$- 0.0000000168$

$= \underline{1.90408}$

$1.904\,08\,226$

ii) Bessel's Formula:

$$y(x) = \frac{y_0 + y_1}{2} + (u - \tfrac{1}{2}) \Delta y_0 + \frac{u(u-1)}{2}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right]$$

$$+ \frac{(u-\tfrac{1}{2})u(u-1)}{6}\left[\Delta^3 y_{-1}\right] + \cdots$$

$$= \frac{1.896481 + 1.915541}{2} + (0.4 - 0.5)(0.01906) +$$

$$\frac{(0.4)(0.4-1)}{2}\left[\frac{0.000189 + 0.000191}{2}\right] + \cdots$$

$$\frac{(0.4-0.5)(0.4)(0.4-1)}{6}\left[0.000002\right]$$

1.906011

$= \cancel{1.896481 + 1.915541} - 0.001906 - 0.0000228$

$+ 0.000000032$

$= \cancel{0.000000008}$

$= \underline{1.904082}$