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UNIT- I

Argument and Entry:

Consider the function, $y = f(x)$. Let x take the values $x_0, x_1, x_2, \ldots x_n$ and y take the values $y_0, y_1, y_2, \ldots y_n$. Alternatively,

x	x_0	x_1	$\ldots$	x_n
y	y_0	y_1	$\ldots$	y_n

The independent variable x is called the argument and the dependent variable y is called the entry

First Difference:

Consider, $y_1 - y_0, y_2 - y_1, y_3 - y_2, \ldots$

These difference are called as first difference of y and is denoted by Δy.

i.e. $\Delta y_0 = y_1 - y_0$

$\Delta y_1 = y_2 - y_1$

$\Delta y_{n-1} = y_n - y_{n-1}$

Δ is called as forward difference operator

Second and higher difference:

$\Delta^2 y_0 = \Delta(\Delta y_0)$

$= \Delta(y_1 - y_0)$

$= \Delta y_1 - \Delta y_0$

$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$

llly.

$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$

Arguments ...

x_0, x_0+h ...

instead of $x_0, x_1, \ldots, x_n$ ...

h is the length of ...

difference.

(1) Forward difference ...

$\Delta f(x) = f(x+h) - f(x)$

(2) Backward difference ...

∇(del) is called as backward difference operator and is defined as,

$\nabla f(x) = f(x) - f(x-h)$

(3) Central ...

$\delta y_{\frac{1}{2}} = \ldots$

Note

$E y_0 = y_1$

$E y_1 = y_2$

(8) ~~Explain that~~

Relationship between the operators.

(1) Between Δ & E

$\Delta f(x) = f(x+h) - f(x)$

$= E f(x) - f(x)$

$\Delta f(x) = f(x)[E-1]$

$\Delta = E - 1$

$\boxed{E = \Delta + 1}$

(2) Between ∇ & E

$\nabla f(x) = f(x) - f(x-h)$

$= f(x) - E^{-1} f(x)$

$= f(x)[1 - E^{-1}]$

$\nabla f(x) = f(x)[1 - E^{-1}]$

$\nabla = 1 - E^{-1}$

$E^{-1} = 1 - \nabla$

$\boxed{E = (1-\nabla)^{-1}}$

Difference Table:

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7
x_0	y_0							
		Δy_0						
x_1	y_1		$\Delta^2 y_0$					
		Δy_1		$\Delta^3 y_0$				
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$			
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$		
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$	
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$		$\Delta^7 y$
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		$\Delta^6 y_1$	
		Δy_4		$\Delta^3 y_3$		$\Delta^5 y_2$		
x_5	y_5		$\Delta^2 y_4$		$\Delta^4 y_3$			
		Δy_5		$\Delta^3 y_4$				
x_6	y_6		$\Delta^2 y_5$					
		Δy_6						
x_7	y_7							

① Form a difference table for the following data

x	y	Δ	Δ²	Δ³	Δ⁴	Δ⁵
1	4					
		11				
			14			
2	15			6		
					0	
		25				
			20			
3	40			6		0
		45				
			26		0	
4	85					
		71		6		
5	156		32			
		103				
6	259					

②

x	y	Δ	Δ²	Δ³	Δ⁴	Δ⁵	Δ⁶	Δ⁷
1	105							
		10						
			255					
2	115			-200				
		265			40			
3	380		55			110		
				-160				
		320					-50	
			-105		150			
4	1700					60		-540
		215		-10			-590	
			-115		210			
5	915			200		-530		
		100			-320			
6	1015		85	-120				
		185						
7	1200		-35					
		150						
8	1350							

③

x	y	Δ	Δ²	Δ³	Δ⁴	Δ⁵	Δ⁶	Δ⁷	Δ⁸
5	13								
		5							
10	18		17						
		22		-4					
15	40		13		-24				
		35		-20		120			
20	75		-7		96				
		28		76			308		
25	103		69		92	-188		464	
		97		-16			156		-48
30	200		53		-124	-32		-18	[illegible]
		150		-140		106	138		-130
35	350		-87		-18			148	
		63		122		96	-10		
40	413		35		78				
		98		144					
45	511		-9						
		89							
50	600								

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To find y_k in terms of y_0, Δy_0, $\Delta^2 y_0, \dots$

$$y_k = E^k y_0$$

$$= (1+\Delta)^k y_0$$

$$= (1 + kC_1 \Delta + kC_2 \Delta^2 + kC_3 \Delta^3 + \dots) y_0$$

$$\therefore y_k = y_0 + kC_1 \Delta y_0 + kC_2 \Delta^2 y_0 + \dots$$

① Find the 6^{th} term of the series

2, 9, 28, 65, 126, 217

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	2	7	12	6	0	0
1	9	19	18	6	0	
2	28	37	24	6		
3	65	61	30			
4	126	91				
5	217					

$$y_6 = (1+\Delta)^6 y_0$$

$$= y_0 + 6C_1 \Delta y_0 + 6C_2 \Delta^2 y_0 + 6C_3 \Delta^3 y_0 + 6C_4 \Delta^4 y_0 +$$
$$6C_5 \Delta^5 y_0 + 6C_5 \Delta^5 y_0 + 6C_6 \Delta^6 y_0$$

$$= 2 + (6\times7) + \frac{6\times5}{1\times2}\times12 + \frac{6\times5\times4}{1\times2\times3}\times6 + 0 + 0$$

$$= 2 + 42 + 180 + 120$$

$$= 344$$

② Find the sixth term of the sequence

x	y	Δy_0	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	8	4	3	0	0
1	12	7	3	0	
2	19	10	3		
3	29	13			
4	42				

$E = 1 + \Delta$

$$y_5 = (1+\Delta)^5 y_0$$

$$= y_0 + 5C_1 \Delta y + 5C_2 \Delta^2 y + 5C_3 \Delta^3 y + 5C_4 \Delta^4 y$$

$$= 8 + 5C_1 4 + 5C_2 3 + 0 + 0$$

$$= \cancel{8 + 5\times}$$

$$= 8 + (5\times4) + \frac{5\times4}{1\times2}\times3$$

$$= 8 + 20 + 30$$

$$= 58$$

(?) Find the first term of the series whose second and subsequent terms are 8, 3, 0, -1, 0, ...

x	y	Δ	Δ²	Δ³	Δ⁴	Δ⁵
0	y_0					
		—				
1	8		—			
		-5		—		
2	3		2		—	
		-3		0		—
3	0		2		0	
		-1		0		
4	-1		2			
		1				
5	0					

$$y_0 = E^{-1} y_1$$

$$= (1+\Delta)^{-1} y_1$$

$$y_0 = (1 - \Delta + \Delta^2 - \Delta^3 + \Delta^4 - \Delta^5) y_1$$

$$y_0 = y_1 - \Delta y_1 + \Delta^2 y_1 - \Delta^3 y_1 + \Delta^4 y_1 - \Delta^5 y_1$$

$$= 8 - (-5) + 2 - 0 + 0 - 0$$

$$= 8 + 5 + 2 = \underline{\underline{15}}$$

(11) In the following table find the missing value.

x	2	3	4	5	6
$f(x)$	45	49.2	54.1	...	67.4

Sol

Only 4 values of $f(x)$ is given

We assume that the polynomial equation is of degree three and 4th differences are zero.

ie) $\Delta^4 y_0 = 0$. $E = 1+\Delta$

$$(E-1)^4 y_0 = 0$$

$$(E^4 - 4C_1 E^3 + 4C_2 E^2 - 4C_3 E + 4C_4 E^0) y_0 = 0$$

$$E^4 y_0 - 4C_1 E^3 y_0 + 4C_2 E^2 y_0 - 4C_3 E y_0 + 4C_4 E^0 y_0 = 0$$

$$y_4 - 4C_1 y_3 + 4C_2 y_2 - 4C_3 y_1 + 4C_4 y_0 = 0$$

$$67.4 - 4(y_3) + 6(\overset{54.1}{\cancel{y_2}}) - 4(49.2) + 45 = 0$$

$$67.4 - 4y_3 + 324.6 - 196.8 + 45 = 0$$

$$\cancel{240.2}\ \cancel{633.8}\ 240.2 = 4y_3$$

$$\therefore y_3 = \cancel{158.45}\ \underline{60.05}$$

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Interpolation and Equal Interval

Interpolation has been described as the art of reading bt the line of a table. It is the prosses of computing intermidiate values of a function from a given set of tabular values of the function. Suppose the following table represents a set of corresponding values of x and y

x	x_0	x_1	...	x_n
y	y_0	y_1	...	y_n

Interpolation is the prosses of finding the value $y = y_i$ corresponding to a value where $x_0 \leq x_i < x_n$

There are two types of interpolation. when the interval x is equal we have interpolation formulas for equal intervals and when the intervals are not equal we have interpolation formulas for unequal intervals.

Interpolation with equal Intervals.

(1) Newton's forward interpolation formula:

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots$$

(2) Newton's Backward interpolation formula:

$$y(x) = y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \dots$$

where

$$v = \frac{x - x_n}{h}$$

① From the following table which gives premium for different ages, Fd. the premium values corresponding to age 46 and 63.

Age	45	50	55	60	65
Premium	114.84	96.16	83.32	74.48	68.48

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
45	114.84				
		-18.68			
50	96.16		5.84		
		-12.84		-1.84	
55	83.32		4		0.68
		-8.84		-1.16	
60	74.48		2.84		
		-6			
65	68.48				

Newton's forward formula:

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \dots$$

$$u = \frac{x - x_0}{h} = \frac{46-45}{5} = 0.2$$

$$y(46) = 114.84 + 0.2(-18.68) + \frac{0.2(0.2-1)}{2} \times 5.84 + \frac{0.2(0.2-1)(0.2-2)}{6} \times -1.84 + \frac{0.2(0.2-1)(0.2-2)(0.2-3)}{24} \times 0.68$$

$$= 114.84 - 3.74 - 0.4676 - 0.0883 - 0.0228$$

$$= 110.5217$$

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$$y(x) = y_n + v\nabla y_n + \frac{v(v+1)}{2}\nabla^2 y_n + \dots$$

$$v = \frac{x - x_n}{h} = \frac{63-65}{5} = -2/5 = -0.4$$

$$y(63) = 68.48 + (-0.4)(-6) + \frac{(-0.4)(0.6)}{2}(2.84) + \frac{(-0.4)(0.6)(1.6)}{6}(-1.16) + \frac{(-0.4)(0.6)(1.6)(2.6)}{24}(0.68)$$

$$= 68.48 + 2.4 - 0.34 + 0.07 - 0.028$$

$$= 70.582$$

(9) Find the value of y at $x = 21$ and $x = 28$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	0.342			
		0.0487		
23	0.3907		-0.001	~~0.0003~~
		0.0477		~~0.0003~~
26	0.4384		-0.0013	-0.0003
		0.0464		
29	0.4848			

When $x = 21$

$$u = \frac{x - x_0}{h} = \frac{21 - 20}{3} = \frac{1}{3} = 0.33$$

$$y(21) = 0.342 + (0.33)(0.0487) + \frac{(0.33)(-0.67)}{2}(-0.001)$$

$$+ \frac{(0.33)(-0.67)(-1.67)}{6}(-0.0003)$$

$$= 0.342 + 0.0160 + 0.00011 - 0.00018$$

$$= \cancel{0.35829} \quad 0.358$$

$$\approx 0.358$$

When $x = 28$

$$v = \frac{x - x_n}{h} = \ldots$$

$$y(28) = 0.4848 + (-0.33)\ldots$$

$$= 0.4848 - 0.0152 + \ldots$$

$$= 0.469[illegible]$$

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From the following [illegible]

x	45	[illegible]	[illegible]
$\tan x$	1.00	[illegible]	[illegible]

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
45	1.0000					
		0.03553				
46	1.03553		0.00131			
		0.03684		0.00009		
47	1.07237		0.0014 ~~0.00689~~		0.00003	
		0.03824 ~~0.04373~~		0.00012		−0.00005
48	1.11061		0.00152 ~~−0.00397~~		−0.00002	
		0.03976		0.0001		
49	1.15037		0.00162			
		0.04138				
50	1.19175					

~~$y(45°15')$~~ =

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)\Delta^2 y_0}{2} + \frac{u(u-1)(u-2)}{6}\Delta^3 y_0$$

$$+ \frac{u(u-1)(u-2)(u-3)}{24}\Delta^4 y_0 + \frac{u(u-1)(u-2)(u-3)(u-4)}{12}\Delta^5 y_0$$

$$= 1 + (0.25)(0.03553) + \frac{(0.25)(-0.75)}{2}(0.00131) +$$

$$\frac{(0.25)(-0.75)(-1.75)}{6}(0.00009) + \frac{(0.25)(-0.75)(-1.75)(-2.5)}{24}(0.00003)$$

$$+ \frac{0.25(-0.75)(-1.75)(-2.75)(-3.75)}{120}(0.00005)$$

$= 1 + 0.0089 - 0.000122 + 0.000005 -$
0.00000112 ~~$+ 0.00000028$~~ $+ 0.00000141$

$=$ ~~1.0087~~ $\underline{1.00878}$

(2) Find the value of y at $x = 1.05$

x	1	1.1	1.2	1.3	1.4	1.5
y	0.841	0.891	0.932	0.964	0.985	0.15

0.8678

(3) Find $f(1.02)$

x	1	1.1	1.2	1.3	1.4
y	1.841	1.891	0.932	0.964	0.985

(4) Find y for $x = 30$

x	21	25	29	33
y	14.27	15.81	17.72	19.96

18.25

(2)

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.0	0.841	0.05	-0.009	0	-0.002	0.02
1.1	0.891	0.041	-0.009	-0.002	0.018	
1.2	0.932	0.032	-0.011	0.02		
1.3	0.964	0.021	0.009			
1.4	0.985	0.03				
1.5	1.015					

$$u = \frac{x - x_0}{h} = \frac{1.05 - 1.0}{0.1} = 0.5$$

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{24}\Delta^4 y_0 + \frac{u(u-1)(u-2)(u-3)(u-4)}{120}\Delta^5 y$$

$y(1.05)$

$$= 0.841 + (0.5)(0.05) + \frac{(0.5)(-0.5)}{2}(-0.009) + \frac{(0.5)(-0.5)(-1.5)}{6}(0) + \frac{(0.5)(-0.5)(-1.5)(-2.5)}{24}(-0.002)$$

$$+ \frac{(0.5)(-0.5)(-1.5)(-2.5)(-3.5)}{120}(0.02)$$

$$= 0.841 + 0.025 + 0.00112 + 0.000078 + 0.00055$$

$$= \underline{0.8677}$$

(3)

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y$	$\Delta^4 y_0$
1.0	1.841	0.05	-1.009	2	-3.002
1.1	1.891	-0.959	0.991	-1.002	
1.2	0.932	0.032	-0.011		
1.3	0.964	0.021			
1.4	0.985				

$$u = \frac{1.02 - 1.0}{0.1} = 0.2$$

$$y(1.02) = 1.841 + (0.2)(0.05) + \frac{(0.2)(-0.8)}{2}(-1.009) + \frac{(0.2)(-0.8)(-1.8)}{6}(2) + \frac{(0.2)(-0.8)(-1.8)(-2.8)}{24}(-3.002)$$

$$= 1.841 + 0.01 + 0.0807 + 0.096 + 0.1008$$

$$= \underline{2.12856}$$

(4)

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
[illegible]	[illegible]			
		1.54		
25	15.81		0.37	
		1.91		-0.04
29	17.72		0.33	
		2.24		
33	19.96			

$$y(30) = y_n + v\nabla y_n + \frac{v(v+1)}{2}\nabla^2 y_n + \frac{v(v+1)(v+2)}{6}\nabla^3 y_n$$

$$v = \frac{x - x_n}{h} = \frac{30 - 33}{4} = \frac{-3}{4} = -0.75$$

$$y(30) = 19.96 + (-0.75)(2.24) + \frac{(-0.75)(0.25)}{2}(0.33) + \frac{(-0.75)(0.25)(1.25)}{6}(-0.04)$$

$$= 19.96 - 1.68 - 0.0309 + 0.001562$$

$$= \underline{18.250662}$$

7) Find the value of f for x = 43 & 84

x	$f(x)$	Δy	$\Delta^2 y$
4[illegible]	184		
		20	
50	204		2
		22	
60	226		2
		24	
70	250		2
		26	
80	276		2
		28	
90	304		

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y$$

$$u = \frac{x - x_0}{h} = \frac{43 - 40}{10} = [illegible]$$

$$y(43) = 184 + (0.3)(20) + \frac{(0.3)(-0.7)}{2}(2)$$

$$= 184 + 6 - 0.[illegible]$$

$$= \underline{18[illegible]}$$

$$y(x) = y_n + v\nabla y_n + \frac{v(v+1)}{2}\nabla^2 y_n$$

$$v = \frac{x - x_n}{h} = \frac{84 - 90}{10} = \frac{-6}{10} = -0.6$$

$$= 304 + (-0.6)(28) + \frac{(-0.6)(0.4)}{2}(2)$$

$$= 304 - 16.8 - 0.24$$

$$= \underline{286.96}$$

(2) Find pressure for c = 142 & 175

°C	pre	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
140	3.685				
		1.169			
150	4.854		0.279		
		1.448		0.047	
160	6.302		0.326		0.002
		1.774		0.049	
170	8.076		0.375		
		2.149			
180	10.225				

$$u = \frac{x - x_0}{h} = \frac{142 - 140}{10} = \frac{2}{10} = 0.2$$

$$y(142) = 3.685 + (0.2)(1.169) + \frac{(0.2)(-0.8)}{2}(0.279)$$

$$+ \frac{(0.2)(-0.8)(-1.8)}{6}(0.047) + \frac{(0.2)(-0.8)(-1.8)(-2.8)}{24}(0.002)$$

$$= 3.685 + 0.2338 - 0.02232 + 0.0022564$$

$$- 0.0000672$$

$$= \underline{3.8986}$$

$$v = \frac{x - x_n}{h} = \frac{175 - 180}{10} = \frac{-5}{10} = -0.5$$

$$y(175) = 10.225 + (-0.5)(2.149) + \frac{(-0.5)(0.5)}{2}(0.375)$$

$$+ \frac{(-0.5)(0.5)(1.5)}{6}(0.049) + \frac{(-0.5)(0.5)(1.5)(2.5)}{24}(0.002)$$

$$= 10.225 - 1.0745 - 0.0468 - 0.00306$$

$$- 0.0000078$$

$$= \underline{9.100562}$$

Gregory - Newton forward interpolation formula

Let $y = f(x)$ denote a function which takes the values $y_0, y_1, \dots y_n$ corresponding to the values $x_0, x_1, \dots x_n$ resp. of x.

Let us suppose that the values of x viz $x_0, x_1, \dots x_n$ are equidistant

That is

$$x_i - x_{i-1} = h \quad . \text{ For } i = 1, 2, \dots n$$

$$\therefore \quad x_i = x_0^{th} \qquad x_2 = x_0 + 2h + \dots$$

$$x_i = x_0^{th}, \quad i = 1, 2 \dots n$$

Let $P_n(x)$ be a polynomial of the n^{th} degree in x such that

$$y_i = f(x_i) = P_n(x_i), \quad i = 0, 1, \dots n$$

Let us assume $P_n(x)$ in the form given below:

$$P_n(x) = a_0 + a_1 (x - x_0)^1 + a^2 (x - x_0)^2 + \dots a_r (x - x_0)^r + \dots a_n (x - x_0)^n \quad \text{— ①}$$

The $(n+1)$ unknowns $a_0, a_1, \dots a_n$ can be found as follows,

$$P_n(x_0) = y_0 = a_0 \quad (\text{setting } x = x_0 \text{ in ①})$$

$$\Delta^r P_n(x) = a_r r! h^r + \text{terms involving}$$

$(x - x_0)$ as a factors — ②

($\because$ the first r terms vanish)

setting $x = x_0$ in ②,

$$\Delta^r P_n(x_0) = a_r r! h^r \quad (\because \text{the other terms in ② vanish})$$

$$\Delta^r y_0 = a_r r! h^r$$

Hence ~~$a_r r! h^r$~~ $a_r = \frac{1}{r! h^r} \Delta^r y_0$

Putting $r = 1, 2, 3 \dots n$ in ③, we get the values of $a_1, a_2, \dots a_n$

$$P_n(x) = y_0 + \frac{(x-x_0)^1}{h}\Delta y_0 + \frac{(x-x_0)^2}{2!h^2}\Delta^2 y_0$$

$$+ \frac{(x-x_0)^r}{r!\,h^r}\Delta^r y_0 + \cdots$$

$$+ \frac{(x-x_0)^n}{n!\,h^n}\Delta^n y_0 \quad \text{—— (4)}$$

$$\frac{(x-x_0)^r}{h^r} = \frac{x-x_0\,(x-x_0-h)(x-x_0-2h)\cdots(x-x_0-\overline{r-1}h)}{}$$

$$\frac{x-x_0}{h}\left[\frac{x-x_0}{h}-1\right]\left[\frac{x-x_0}{h}-2\right]\cdots\left[\frac{x-x_0}{h}-\overline{r-1}\right]$$

$$= u(u-1)(u-2)\cdots(u-\overline{r-1}), \quad u = \frac{x-x_0}{h}$$

$= u^r$, (here $h=1$) and $x = x_0 + uh$

Using this in (4)

$$P_n(x) = P_n(x_0+h) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u^2}{2!} + \cdots$$

$$\cdots \frac{u^r}{r!}\Delta^r y_0 + \cdots \frac{u^n}{n!}\Delta^n y_0 \quad \text{—— (5)}$$

where $u^r = u(u-1)\cdots(u-r+1)$

If x is given, u is found out.

Eqn (5) is known as Gregory-Newton forward interpolation formula.

__Aliter:__

We can also prove the above formula using symbolic operator method.

$$P_n(x) = P_n(x_0+uh) = E^u P_n(x_0) = E^u y_0$$

$$= (1+\Delta)^u y_0$$

$$= \left[1 + \binom{u}{1}\Delta + \binom{u}{2}\Delta^2 + \binom{u}{3}\Delta^3 + \binom{u}{r}\Delta^r + \cdots \binom{u}{n}\Delta^n\right]$$

$$= y_0 + \frac{u}{1}\Delta y_0 + \frac{u^2}{2!}\Delta^2 y_0 + \frac{u^3}{3!}\Delta^3 y_0 + \cdots$$

$$\frac{\Delta^r}{r!}\Delta^r y_0 + \cdots \frac{\Delta^n}{n!}\Delta^n y_0$$

where

$$u = \frac{x-x_0}{h}.$$

If ~~y(x) is~~ $y(x)$ is a polynomial of n^{th} degree $\Delta^{n+1} y_0, \ldots$ are zero.

Hence

$$P(x) = P_n(x_0 + uh) = y_0 + \frac{u}{1}\Delta y_0 + \frac{u^2}{2!}\Delta^2 y_0 + \cdots \frac{u^n}{n!}\Delta^n y_0$$

Gregory - Newton Backward interpolation formula:

Newton's ~~format~~ forward interpolation formula cannot be used for interpolating a value of y nearer to the end of the table of values. For this purpose we get another backward interpolation formula

Suppose $y = f(x)$ takes the values $y_0, y_1, \ldots y_n$ corresponding to the values $x_0, x_1, \ldots x_n$ of x

Let $x_i - 1 - h$, for $i = 1, 2, \ldots n$ (equal interval)

$\therefore x_i = x_0 + ih$, $i = 0, 1, 2 \ldots n$

Now, we want to find a collacation polynomial $P_n(x)$ of degree n in x such that

$$P_n(x_i) = y_i, \quad i = 0, 1, 2, \ldots n$$

Let

$$P_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n) + \cdots a_r(x - x_n)(x - x_{n-1}) \cdots (x_r - x_{n-r+1}) + \cdots a_n(x - x_n)(x - x_{n-1}) \cdots (x - x_1) \quad (1)$$

$\because x_{n-1} = x_n - h,\ x_{n-2} = x_n - 2h \ldots x_n - rh$

$= x_n - (r-1)h$

$x_1 = x_n - (n-1)h$, we have

$$P_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_n + h) + a_3(x - x_n)(x - x_n + h)(x - x_n + 2h) + \cdots + a_n(x - x_n)(x - x_n + h) \cdots (x - x_n + \overline{n-1}h)$$

$$P_n(x) = a_0 + a_1(x - x_n)^1 + a_2(x - x_n + h)^2 + a_3(x - x_n + 2h)^3 + a_r(x - x_n + \overline{r-1}\,h)^r + \cdots a_n(x - x_n + n - 1h)^n \quad (2)$$

We shall find $a_0, a_1 \ldots a_n$ such that

$P_n(x_i) = y_i$

$\nabla = E^{-1}\Delta$

$\nabla^r (x-a)^m = E^{-r}\Delta^r (x-a)^m$

$= E^{-r}[m(m-1)(m-2)\ldots(m-r-1)h^r (x-a)^{m-r}]$

$= m(m-1)(m-2)\ldots(m-r+1)h^r (x-rh-a)^{m-r}$, if $r \le m$ ——③

$x - x_n$ is a factor in all terms of RHS of ② except in a_0

~~Putting~~ Put $x = x_n$ in ②

$P_n(x_n) = y_n = a_0$

Operating ② by ∇^r using ③

$\nabla^r P_n(x_n) = 0 + 0 + \quad 0 + a_r\, r!\, h^r(r+1)\, r.$
$(r-1)\ldots 2h^r a_{r+1}(x - x_n)'$ terms

involving $(x - x_n)$ as a factor

Setting $x = x_n$ in this,

$\nabla^r P_n(x_n) = \nabla^r y_n = a_r\, r!\, h^r$

($\because$ other items vanish)

$\therefore a_r = \frac{1}{r!h^r}\nabla^r y_n$, when $r = 1, 2, \ldots n$

Putting the values of $a_0, a_1 \ldots a_n$ in ②

we get

$P_n(x) = y_n + \frac{(x-x_n)}{1!h}\nabla y_n + \frac{(x-x_n+h)^2}{2!h^2}\nabla^2 y_n +$

$\frac{(x-x_n+\overline{r-1}\,h)^r}{r!\,h^r}\nabla^r y_n + \ldots$

$\frac{(x-x_n+\overline{n-1}\,h)^n}{n!\,h^n}\nabla^n y_n$ ——⑤

Let $\frac{x-x_n}{h} = v \Rightarrow x = x_n + vh$

Then,

$$\frac{(x - x_n + \overline{r-1}\,h)^r}{h^r} = \frac{(vh + \overline{r-1}\,h)^r}{h^r} = (v + \overline{r-1})^r$$

$\therefore$ (5) becomes,

$$P_n(x) = P_n(x_n + vh) = y_n + \frac{v}{1!}\nabla y_n + \frac{(v+1)^2}{2!}\nabla^2 y_n + \dots$$

$$\frac{(v+r-1)^r}{r!}\nabla^r y_n + \dots \frac{(v+n-1)^n}{n!}\nabla^n y_n.$$

$\therefore P_n(x) = P_n(x_n + vh)$

$$= y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n$$

$$+ \dots \frac{v(v+1)(v+2)+\dots(v+n-1)}{n!}\nabla^n y_n \quad \text{--- (6)}$$

This equation is known as Newton's backward difference interpolation formula.

Aliter:

We can also derive the above formula by symbolic operator methods

$$P_n(x) = P_n(x_n + vh) = E^v P_n(x_n)$$

$$= (1-\nabla)^{-v} y_n \qquad \because E = (1-\nabla)^{-1}$$

$$= \left[1 + v\nabla + \frac{v(v+1)}{2!}\nabla^2 + \frac{v(v+1)(v+2)}{3!}\nabla^3 + \dots\right] y_n$$

$$P_n(x) = P_n(x_n + vh)$$

$$= y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \dots \quad \text{--- (7)}$$

Where,

$$v = \frac{x - x_n}{h}.$$

19/12

① Fd. the following data fd. the no. of students whose weight is bet. ~~60 to 70~~ 40 to 50

$\therefore x = 50$

Weight	0-40	40-60	60-80	80-100	100-120
No. of stud	250	120	100	70	50

Below x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	250				
		120			
60	370		-20		
		100		-10	
80	470		-30		20
		70		10	
100	540		-20		
		50			
120	590				

$$f(x) = y_0 + u(\Delta y_0) + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y_0$$

$$u = \frac{x - x_0}{h} = \frac{50-40}{20} = \frac{10}{20} = 0.5$$

$$f(50) = 250 + 0.5(120) + \frac{(0.5)(-0.5)}{2}(-20) + \frac{(0.5)(-0.5)(-1.5)}{6}(-10) + \frac{(0.5)(-0.5)(-1.5)(-2.5)}{24}(20)$$

$= 250 + 60 + 25 - 0.625 - 0.78125$

$= 311.09375$

Students bet 40-50

$=$ Below 50 $-$ Below 40

$= \cancel{370 - 311}$ $\quad 311 - 250$

$= \cancel{59}$ $\quad = 61$

② Fd. the value of $f(x)$ at $x = 9$

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
2	94.8			
		-6.9		
5	87.9		0.3	
		-6.6		0.1
8	81.3		0.4	
		-6.2		
11	75.1			

$$V = \frac{x - x_n}{h} = \frac{9-11}{3} = -\frac{2}{3} = -0.67$$

$$f(x) = y_n + v\nabla y_n + \frac{v(v+1)}{2}\nabla^2 y + \frac{v(v+1)(v+2)}{6}\nabla^3 y$$

$$= 75.1 + (-0.67)(-6.2) + \frac{(-0.67)(0.33)}{2}(0.4) + \frac{(-0.67)(0.33)(1.33)}{06}(0.1)$$

$= 75.1 + 4.154 + 0.04422 - 0.004901$

$= 79.294279$

3) ...

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	0.3413				
		0.0118			
1.05	0.3531		−0.0006		
		0.0112		0	
1.10	0.3643		−0.0006		0
		0.0106		0	
1.15	0.3749		−0.0006		
		0.01			
1.20	0.3849				

$$u = \frac{1.02-1}{0.05} = \frac{0.02}{0.05} = 0.4$$

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u+1)}{2}\Delta^2 y_0$$

$$= 0.3413 + 0.4(0.0118) + \frac{0.4(1.4)}{2}(-0.0006)$$

$$= 0.3413 + 0.00472 - 0.000168$$

$$= \underline{0.341604}$$

Find tan 0.12 and 0.26

x	0.10	0.15	0.20	0.25	0.3
$\tan x$	0.1003	0.1511	0.2027	0.2553	0.3093

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.001		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.054			
0.3	0.3093				

i) $u = \frac{0.12-0.1}{0.05} = \frac{0.02}{0.05} = 0.4$

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u+1)}{2}\Delta^2 y_0 + \frac{u(u+1)(u+2)}{6}\Delta^3 y_0 + \frac{u(u+1)(u+2)(u+3)}{24}\Delta^4 y_0$$

$$= 0.1003 + (0.4)(0.0508) + \frac{(0.4)(1.4)}{2}(0.0008) + \frac{(0.4)(1.4)(2.4)}{6}(0.000[illegible])$$

$$+ \frac{(0.4)(1.4)(2.4)(3.4)}{24}(0.0002)$$

$$= 0.1003 + 0.02032 + 0.000224 + 0.0000448 + 0.00003808$$

$$= 0.03065688$$

$= 75.1 + 4.154 + 0.04422 - 0.004901$

$= 79.2941279$

(3) fd $f(1.02)$

x	y	Δy	$\Delta^2 y$		
1	0.3413				
		0.0118			
1.05	0.3531		-0.0006		
		0.0112		0	
1.10	0.3643		-0.0006		0
		0.0106		0	
1.15	0.3749		-0.0006		
		0.01			
1.20	0.3849				

$u = \frac{1.02 - 1}{0.05} = \frac{0.02}{0.05} = 0.4$

$f(x) = y_0 + u\Delta y_0 + \frac{u(u+1)}{2}\Delta^2 y_0$

$= 0.3413 + 0.4(0.0118) + \frac{0.4(1.4)}{2}(-0.0006)$

$= 0.3413 + 0.00472 - 0.000168$

$= \underline{0.341604}$

(4) Fd. tan 0.12 and 0.26

x	0.1	0.15	0.20	0.25	0.3
$\tan x$	0.1003	0.1511	0.2027	0.2553	0.3093

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.1	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.001		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.054			
0.3	0.3093				

i) $u = \frac{0.12 - 0.1}{0.05} = \frac{0.02}{0.05} = 0.4$

$f(x) = y_0 + u\Delta y_0 + \frac{u(u+1)}{2}\Delta^2 y_0 + \frac{u(u+1)(u+2)}{6}\Delta^3 y_0 + \frac{u(u+1)(u+2)(u+3)}{24}\Delta^4 y_0$

$= 0.1003 + (0.4)(0.0508) + \frac{(0.4)(1.4)}{2}(0.0008) + \frac{(0.4)(1.4)(2.4)}{6}($

$+ \frac{(0.4)(1.4)(2.4)(3.4)}{24}(0.0002)$

$= 0.1003 + 0.02032 + 0.000224 + 0.00004$

$+ 0.00003808$

$= 0.03065688$

ii) $v = \frac{0.26 - 0.3}{0.05} = -0.8$

$f(x) = y_n + v\nabla y_n + \frac{v(v+1)}{2}\nabla^2 y_n + \frac{v(v+1)(v+2)}{6}\nabla^3 y_n$

$+ \frac{v(v+1)(v+2)(v+3)}{24}\nabla^4 y_n$.

$0.3093 + (-0.8)(\overset{0.054}{\cancel{0.30983}}) + \frac{(-0.8)(-1.8)}{2}(0.0014)$

$+ \frac{(-0.8)(-1.8)(-2.8)}{6}(0.0004) +$

$\frac{(-0.8)(-1.8)(-2.8)(-3.8)}{24}(0.0002)$

$= 0.3093 - 0.0432 + 0.001008 - 0.0002688$
$+ 0.00012768$

$= \underline{\underline{0.26696688}}$

⑤ i) $u = \frac{0.11 - 0.1}{0.05} = \frac{0.01}{0.05} = 0.2$

$f(0.11) = 0.1003 + (0.2)(0.0508) + \frac{(0.2)(1.2)}{2}(0.0008)$

$+ \frac{(0.2)(1.2)(2.2)}{6}(0.0002)$

28/12

①

Year :	1917	1918	1919	1920	1921
Export :	443	384	-	397	467

∵ four values are given we make an assumption that we get a third degree polynomial.
Hence fourth differences of $P_n(x)$ are 0
Let us assume that

$u_0 = 443$, $u_1 = 384$, $u_2 = ?$, $u_3 = 397$
$u_4 = 467$.

Sol:

$\Delta^4 u_0 = 0$

$(E-1)^4 u_0 = 0$

$(E^4 - 4C_1 E^3 + 4C_2 E^2 - 4C_3 E + 1)u_0 = 0$

$E^4 u_0 - 4E^3 u_0 + 6E^2 u_0 - 4Eu_0 + u_0 = 0$

$u_4 - 4u_3 + 6u_2 - 4u_1 + u_0 = 0$

$467 - 4(397) + 6(u_2) - 4(384) + 443 =$

$-2214 \; 808 + 6u_2 = 0 \Rightarrow u$

(2) x : 0 1 2 3 4

y : 1 2 4 - 16

Sol:

$P_n(x) = 0$

Third Degree polynomial.

$\therefore \cancel{E} \Delta^4 y u_0 = 0$

$(E-1)^4 u_0 = 0$

$E^4 u_0 - 4E^3 u_0 + 6E^2 u_0 \mp 4E u_0 + u_0 = 0$

$u_4 - 4u_3 + 6u_2 \mp 4u_1 + u_0 = 0$

$16 - 4(u_3) + 6(4) \mp 4(2) + 1 = 0$

$16 - 4u_3 + 24 \mp 8 + 1 = 0$

$33\cancel{49} = 4u_3$

$\underline{u_3 = \cancel{12.25}\ 8.25}$