MATHEMATICS OF FINANCE

1. Basic Concepts

There are large number of causes for an individual or a corporate body or a Government to borrow. The lender is called the 'creditor' and the borrower is called the 'debtor'. The amount borrowed by the debtor is called the **principal**. The 'fees' or the 'charge' paid by the borrower for using the money of the lender is called **interest**. Borrower has to repay the principal together with interest to the lender. This total amount to be repaid is called the **amount**. Hence, amount is principal plus interest. **Unit of time** is usually one year. Under one method of interest calculation, viz., compound interest, month, quarter year, half year, etc. are also taken as units of time. **Rate of Interest** is the interest for Rs.100 per unit of time.

In other contexts such as depositing surplus money in banks for safety and income (interest), delayed payment while buying goods on credit, etc. also interest is paid by the party who owes some amount to the other.

2. Simple Interest and Compound Interest

Simple Interest and Compound Interest are calculated as follows:

Under simple interest, the interest for one unit of time (usually, one year) is multiplied by the number of units of time to determine the interest due. For example, if Rs.600 is the interest for one year, Rs.1,800 (600x3) is the interest for three years (Refer Example 1). It is to be noted that interests at various units of time are equal under this method. In example 1.

interest for the first year = interest for the second year

= interest for the third year

= Rs. 600.

Under compound interest, the interest for the first unit of time is equal to the simple interest per unit of time. The interest for the second unit of time is to be calculated on the original principal and the interest of the first unit of time. Hence, it will be more than that of the first unit of time. For the calculation of the interest for the third unit of time, the original principal, the interest for the first unit of time and the interest for the second unit of time are to be added and the total is to be taken as the principal. This procedure may be repeated as many times as necessary. The same are obtained by the formulae given later. It is to be noted that the interest for the first unit of time is less than that of the second, the interest for the second unit of time is less than that of the third, etc. Under this method, interest earns interest.

3. Symbols Used

P - Principal or present value

A - Amount or value at the end

n - Number of units of time and n is either an integer or a fraction.

r - Rate of interest (for Rs, 100 per unit of time)

$i = r/100$

s - Effective rate of interest under C.I. which is the interest per Rs.100 per annum when there are m (>1) units of time in one year.

I - Simple interest

CI - Compound interest

4. Simple Interest - Formulae and Problems

Formulae:

(i) $I = \frac{Pnr}{100}$ (ii) $A = P + I = P + \frac{Pnr}{100} = P\left(1 + \frac{nr}{100}\right)$

(iii) $P = \frac{100\,I}{nr}$ (iv) $n = \frac{100\,I}{Pr}$ (v) $r = \frac{100\,I}{Pn}$

Unit of time is 1 year. n is either an integer or a fraction.

Example 1: Find the simple interest on the sum of Rs.6,000 at 10% p.a. for 3 years. (B.Com. Bharathiar, N 93)

Solution: Given: P = 6,000; r = 10; n = 3.

$$\text{Required simple interest, } I = \frac{Pnr}{100} = \frac{6000 \times 3 \times 10}{100}$$

$$= \text{Rs. } 1,800$$

Note: 1. r = 10% p.a. means 10 per cent per annum. That is, interest is Rs.10 per annum for a loan (principal) of Rs.100.

$\therefore$ for Rs.6,000, interest $= \frac{10}{100} \times 6000 = \text{Rs.}600$ per annum.

$\therefore$ interest for 3 years $= 600 \times 3 = \text{Rs. } 1,800$

2. The same may also be found as Simple Interest for one year $= \frac{Pr}{100} = \frac{6000 \times 10}{100} = \text{Rs. } 600$. Simple Interest for three years $= \left(\frac{Pr}{100}\right)n = \text{Rs.}600 \times 3 = \text{Rs.}1,800$.

3. In most of the following problems, formulae alone are used for solving them.

4. If amount is needed, it would be calculated as follows:

Amount, A = P+I = Rs. 6,000+Rs.1,800 = Rs. 7,800.

Example 2: Calculate the total amount that will be received from the debtor when the principal Rs.10,000 is lent to him on interest for 4 years at 9% p.a.

Solution: Given: P = 10,000; n = 4; r = 9.

$$\text{Required amount, } A = P\left(1 + \frac{nr}{100}\right) = 10000\left(1 + \frac{4 \times 9}{100}\right)$$

$$= \text{Rs. } 13,600$$

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Example 3 : Rs.6,000 amounts to Rs. 8,940 at 14% p.a. interest. Find the number of years for which the amount was lent.

Solution : Given: P = 6,000 ; A = 8,940; r = 14.

∴ Interest, I = A − P = 2,940

By substituting in the formula, $n = \frac{100\,I}{P\,r}$, the number of years, $n = \frac{100 \times 2940}{6000 \times 14} = 3.5$ years.

Example 4: If a term deposit of Rs.4,000 earns an interest of Rs.2,500 in 50 months, find the rate of interest.

Solution: Given: P = 4,000; I = 2,500; $n = \frac{50}{12} = 4\frac{2}{12}$ years. Using the formula, $r = \frac{100\,I}{P\,n}$

the rate of interest, $r = \frac{100 \times 2500}{4000 \times \frac{50}{12}} = \frac{100 \times 2500 \times 12}{4000 \times 50} = 15\%$ p.a.

Example 5: A certain sum amounts to Rs.4,000 at the end of 5 years at 12% p.a. interest. Find the sum.

Solution: Given : A = 4,000; n = 5; r = 12.

Required to find P. Consider, $A = P\left(1 + \frac{n\,r}{100}\right)$

Rewriting and then substituting, required sum (principal),

$$P = \frac{A}{1 + \frac{n\,r}{100}} = \frac{4000}{1 + \frac{5 \times 12}{100}} = \frac{4000}{1.6} = \text{Rs. } 2{,}500$$

Example 6: Mr. Ramesh deposited Rs.25,000 on 1.1.94. At the end of 5 months, he withdrew Rs.5,000. Find the interest due to him on 31.12.94. Rate of interest = 12% per annum.

(B.B.M. Bharathiar, A96)

Solution: Interest for Rs.25,000 for 5 months

$$= \frac{P\,n\,r}{100} = 25{,}000 \times \frac{5}{12} \times \frac{12}{100} = \text{Rs. } 1{,}250$$

Interest for Rs.20,000 for 7 months

$$= 20{,}000 \times \frac{7}{12} \times \frac{12}{100} \qquad = \text{Rs.}1{,}400$$

Interest due on 31.12 94 = Rs.1,250 + Rs.1,400 = Rs.2,650

Example 7: A sum of money amounted to Rs.1,071 in 6 months and Rs.1,106 in 16 months. Calculate the rate of simple interest. (B.Com. Bharathiar, N 83; B.Com. Madras S90; B.Com./B.C.S. Bharathiar, N99)

Solution: Amount at the end of 16 months = 1,106

Amount at the end of 6 months = 1,071

$\therefore$ Interest for (16–6) = 10 months = 1106–1071 = 35

$\therefore$ Interest for 6 months $= \frac{35}{10} \times 6 = 21$

Hence, Principal = Amount–Interest = 1071–21 = 1050

$\therefore$ Rate of simple interest, $r = \frac{100\,I}{P\,n} = \frac{100 \times 21}{1050 \times \frac{6}{12}} = \frac{2100}{525}$

$= 4\%$

Example 8: Mr. Somasundaram deposits a total of Rs.45,000 in two different banks which give 10% and 15% interest respectively. If the amounts repayable by the two banks at the end of 10 years are to be equal, determine the individual amounts of deposit.

Solution: Let the amount deposited in the first bank be Rs. x. i.e., Principal, P = x

$$= \frac{}{n\,r} = \frac{}{3 \times 12}$$

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5 Compound Interest - Formulae and Problems

Usually rate of interest is quoted per annum. When unit of time is different from a year, the corresponding units of time are to be identified and taken as n and the rate of interest per unit of time is to be taken as r.

For example, when compound interest is to be calculated for 5 years at 8% per annum compounded quarterly (Example15), as there are 4 quarter years in 1 year, $r = \frac{8}{4} = 2$ and $n = 5 \times 4 = 20$. Interest calculated on quarterly basis will be more than that calculated on half yearly basis; interest calculated on half yearly basis will be more than that calculated on yearly basis. As interest earns interest, interest calculated more times will naturally be more when other things remain the same.

Formulae:

(i) $A = P\left(1 + \frac{r}{100}\right)^n$ (ii) $CI = A - P = P\left[\left(1 + \frac{r}{100}\right)^n - 1\right]$

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(iii) $P = \dfrac{A}{\left(1+\dfrac{r}{100}\right)^n}$ (iv) $r = 100\left[\left(\dfrac{A}{P}\right)^{\frac{1}{n}} - 1\right]$

(v) $n = \dfrac{\log A - \log P}{\log\left(1+\dfrac{r}{100}\right)}$ (Or)

(i) $A = P(1+i)^n$ (ii) $C.I. = A-P = P[(1+i)^n-1]$

(iii) $P = \dfrac{A}{(1+i)^n}$ (iv) $i = \left(\dfrac{A}{P}\right)^{\frac{1}{n}} - 1$ (v) $n = \dfrac{\log A - \log P}{\log(1+i)}$

Note: n is either an integer or a fraction.

Example 14: Find the compound interest on Rs.20,000 for 5 years at 20% per annum. What will be the simple interest in the above case? (B.Com. Bharathiar. A96)

Solution: Given : $P = 20{,}000$, $n = 5$; $r = 20$.

$\therefore$ Compound Interest, CI $= P\left[\left(1+\dfrac{r}{100}\right)^n - 1\right]$

$= 20000\left[\left(1+\dfrac{20}{100}\right)^5 - 1\right] = 20000[(1.2)^5-1]$

$=$ Rs. 29,766.40

Further, Simple Interest, $I = \dfrac{P\,n\,r}{100} = \dfrac{20000 \times 5 \times 20}{100}$

$=$ Rs. 20,000

Note :1. Compound interest is more than the simple interest.

2. Compound interest may also be calculated as follows (without using the formula):

Principal for the first year $= 20000$

Interest for the first year $= \dfrac{P\,n\,r}{100} = \dfrac{20000 \times 1 \times 20}{100} = 4{,}000$

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Principal for the second year = 20,000 + 4000 = 24,000

Interest for the second year = $\frac{24000 \times 1 \times 20}{100}$ = 4,800

Principal for the third year = 24000 + 4800 = 28,800

Interest for the third year = $\frac{28800 \times 1 \times 20}{100}$ = 5,760

Principal for the fourth year = 28800 + 5760 = 34,560

Interest for the fourth year = $\frac{34560 \times 1 \times 20}{100}$ = 6,912

Principal for the fifth year = 34560 + 6912 = 41,472

Interest for the fifth year = $\frac{41472 \times 1 \times 20}{100}$ =8294.40

Amount to be repaid at the end of five years,

A = 41,472 + 82,94.40 = Rs. 49,766.40

Compound Interest = A–P = 49,766.40 – 20,000

= Rs. 29,766.40

If n is a fraction, for the last unit of time 'n' will not be 1 but will be a fraction in $\frac{Pnr}{100}$.

This procedure may not fetch credit in the examinations.

3. Under the above note, it is seen that compound interest for first year < compound interest for second year < compound interest for third year < .....

4. The formula for A can be derived as shown below

Principal for the first unit of time = P, say

Interest for the first unit of time = $\frac{P \times 1 \times r}{100} = \frac{Pr}{100}$

Amount due at the end of first unit of time which is the principal for the second unit of time = $P + \frac{Pr}{100} = P\left(1 + \frac{r}{100}\right)$

Interest for the second unit of time $= \dfrac{P\left(1+\frac{r}{100}\right)\times 1\times r}{100}$

$$= P\left(1+\frac{r}{100}\right)\frac{r}{100}$$

Amount due at the end of second unit of time which is the principal for the third unit of time

$$= P\left(1+\frac{r}{100}\right) + P\left(1+\frac{r}{100}\right)\frac{r}{100}$$

$$= P\left(1+\frac{r}{100}\right)\left[1+\frac{r}{100}\right] = P\left(1+\frac{r}{100}\right)^2$$

As we proceed in this manner, we would get the amount due at the end of n years as $A = P\left(1+\frac{r}{100}\right)^n$

Example 15 (i): Calculate the compound interest for Rs.2,500 for 4 years at 8% per annum. (B.Com. Bharathiar, N96)

(ii) Calculate the compound interest in the above case when interest is compounded (a) half yearly and (b) quarterly.

Solution : (i) Given : P = 2500; n = 4; r = 8.

Compound Interest, $CI = P\left[\left(1+\frac{r}{100}\right)^n - 1\right]$

$$= 2500\left[\left(1+\frac{8}{100}\right)^4 - 1\right] = \text{Rs. } 901.22$$

(ii) (a) P = 2500; n = 8 (half years) ; $r = \frac{8}{2} = 4$

Compound interest, $CI = 2500\left[\left(1+\frac{4}{100}\right)^8 - 1\right] = \text{Rs. } 921.42$

(b) P = 2500; n = 16 (quarter years) ; $r = \frac{8}{4} = 2$

Compound interest, $CI = 2500\left[\left(1+\frac{2}{100}\right)^{16} - 1\right] = \text{Rs.} 931.96$

Note : 1. Compound interest calculated quarterly is more than that calculated half yearly. Compound interest calculated half yearly is more than that calculated yearly.

Example 16: Mr. Ambigapathy borrows Rs.1,00,000 at 24% compounded monthly. Find the amount he has to repay at the end of 3 years.

Solution: Given:P = 1,00,000; $r=\frac{24}{12}=2$;

$$n = 3 \times 12 = 36 \text{ (months)}.$$

The amount he has to repay,

$$A = P\left(1+\frac{r}{100}\right)^n = 100000\left(1+\frac{2}{100}\right)^{36} = \text{Rs.}2{,}03{,}988.73$$

Example 17: Balu borrowed Rs.25,000 from Rathinam but could not repay the amount in a period of 5 years. Accordingly, Rathinam demands now Rs.35,880 from Balu. At what percent p.a. compound interest did Rathinam lend his money? (B.Com. Bharathiar, N95)

Soultion: Given : P = 25000; n=5; A = 35,880

By substituting in the formula,

$$r = 100\left[\left(\frac{A}{P}\right)^{\frac{1}{n}} - 1\right] = 100\left[\left(\frac{35880}{25000}\right)^{\frac{1}{5}} - 1\right] = 7.49\% \text{ p.a.}$$

Example 18: V.P. Balaraman deposits Rs.12,000 in Pandurangan Associates and gets Rs. 27,566.93 at the end of $3\frac{1}{2}$ years. Find the rate of compound interest which the company pays per month.

Solution: Given: P = 12000; A = 27,566.93;

$$n = 3\frac{1}{2} \times 12 = 42 \text{ months}$$

For finding the monthly rate of interest, $\frac{r}{12}$, consider

$$r = 100\left[\left(\frac{A}{P}\right)^{\frac{1}{n}} - 1\right]$$

$$\therefore \quad \frac{r}{12} = 100\left[\left(\frac{27566.93}{12000}\right)^{\frac{1}{42}} - 1\right]$$

$$= 2$$

$$\therefore \quad r = 24\% \text{ p.a. paid monthly.}$$

Example 19: What amount lent at 10% p.a. compound interest will fetch Rs.630 as interest in 2 years?

(B.Com. Bharathiar. A94)

Solution: Given: r=10; CI = Rs.630; n=2

To find P, use the formula, $P = \frac{CI}{\left(1+\frac{r}{100}\right)^n - 1} = \frac{630}{\left(1+\frac{10}{100}\right)^2 - 1}$

$= \text{Rs.}3{,}000.$

Example 20: A certain sum deposited in a bank at 15% p.a. compounded monthly amounts to Rs.42,143.63 at the end of 5 years. Find the principal

Solution: Given: $r = \frac{15}{12} = 1.25$; A=42,143.63; n=5x12=60

For finding the principal, P, by substituting in the formula,

$$P = \frac{A}{\left(1+\frac{r}{100}\right)^n} = \frac{42{,}143.63}{\left(1+\frac{1.25}{100}\right)^{60}} = \text{Rs.}20{,}000$$

Example 21: Find the period in which an amount gets DOUBLED at 12% per annum compound interest.

(B.B.M. Bharathiar, A95 and A96)

Solution: Given: A = 2P ; r = 12

B's share = Rs.40,000 − Rs.17,429 = Rs. 22,571

Effective Rate and Nominal Rate of Interest.

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From Example 15 it is known that more often the interest is compounded, greater is the interest earned. It is found that the interest compounded quarterly (4 times in a year) is greater than the interest compounded half yearly (2 times in a year) which is greater than the interest compounded yearly (once in a year). Although the rate of interest is quoted commonly as 8% per annum in the example, it does not result in equal amounts of interest in the three cases.

r, the rate of interest usually quoted per annum, is known as the nominal rate. The actual annual rate of increase in the amount when interest is calculated more than once in a year is known as effective rate. The effective rate is denoted by s.

When compound interest is calculated yearly, r=s. When compound interest is calculated by considering some other unit of time (such as half yearly, quarterly, etc. each of which is less than one year), r < s.

For example, consider P = 100; r = 8% (p.a.) n = 1(year).

$$CI = P\left[\left(1+\frac{r}{100}\right)^n - 1\right] = 100\left[\left(1+\frac{8}{100}\right)^1 - 1\right] = 8$$

In the same situation, if the interest is compounded quarterly, we have P=100; $r=\frac{8}{4}$ = 2 per quarter year and n=1×4=4 quarter years in a year.

$$\therefore CI = P\left[\left(1+\frac{r}{100}\right)^n - 1\right] = 100\left[\left(1+\frac{2}{100}\right)^4 - 1\right] = 8.24$$

We find that r=8% p.a. and s=8.24% p.a.

It may be noted that the compound interest of r% p.a. compounded m times a year (=4 in this example) is equivalent to s% p.a. compounded yearly.

Corresponding to a nominal rate of compound interest of r% p.a., the effective rate of compound interest of s% p.a. can be determined as follows:

The effective rate of compound interest percent per annum (s) is the actual compound interest percent per annum and so, $s = 100\left[\left(1+\frac{r}{100m}\right)^m - 1\right]$ where there are m units of time in one year (i.e. quarterly means m=4, half yearly means m=2, etc.)

The above relation can be used as a formula to find s from r and m.

Example 29: Find the effective rate of interest percent per annum equivalent to a nominal rate 12% per annum, the interest being payable half yearly. (B.B.M. Bharathiar, A93)

Solution: Given r=12; m=2 (half years in one year).

For finding the effective rate of interest percent per annum, s, the following formula is used.

$$s = 100\left[\left(1+\frac{r}{100m}\right)^m - 1\right]$$

∴ by substituting the given values,

$$s = 100\left[\left(1+\frac{12}{100\times 2}\right)^2 - 1\right]$$

$$= 12.36\%\ \text{p.a}$$

Note: Interest is compounded continuously means m is very large. That is, $m \to \infty$. Then

$$s = \lim_{m \to \infty} 100\left[\left(1 + \frac{i}{m}\right)^m - 1\right] = 100\,(e^i - 1)$$

When interest is compounded m times a year, $A=P(1+i)^n$ becomes $A = P\left(1 + \frac{i}{m}\right)^{mn}$.

Hence, when interest is compounded continuously

$$A = \lim_{m \to \infty} P\left(1 + \frac{i}{m}\right)^{mn} = Pe^{in}$$

Consequently, $C.I. = A-P = P(e^{in}-1)$

$$P = \frac{A}{e^{in}} \text{ and}$$

$$n = \frac{\log A - \log P}{i \log e}$$

Example 30: Find the effective rate of interest equivalent to a nominal rate of 12% p.a., compounded monthly.

Further, find the effective rate when interest is compounded continuously.

Solution: By substituting r=12 and m=12 in the formula,

$$s = 100\left[\left(1 + \frac{r}{100m}\right)^m - 1\right],$$

$$s = 100\left[\left(1 + \frac{12}{100 \times 12}\right)^{12} - 1\right] = 12.68\% \text{ p.a.}$$

When interest is compounded continously,

$$s = 100(e^i-1)$$

∴ the required $s = 100(e^{0.12}-1) = 12.75\%$ p.a.

Example 31: If interest is compounded continuously at 6%, in how many years will an amount A double in value?

(B.Com. Madras, N80)

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Solution: By substituting 0.06 in the place of i (=r/100) A in the place of P and 2A in the place of A in

$$n = \frac{\log A - \log P}{i \log e}$$

$$n = \frac{\log 2A - \log A}{0.06 \log e}$$

$$= \frac{\log 2}{0.06 \log e} \qquad \log 2A = \log 2 + \log A$$

$$= \frac{0.3010}{0.06 \times 0.4343} = 11.55 \text{ years.}$$

Example 32: A bank pays interest by continuously compounding interest to investors. A man invests Rs.20,000 in the bank and accrued interest is 15% per year compounded continuously. How much will he get after 10 years?

(B.Com./B.C.S. Bharathiar, N2000)

Solution: By substituting P = 20000, i = 0.15 and n = 10 in the formula, $A = P\, e^{in}$.

$$A = 20000\, e^{0.15 \times 10}$$

$$= \text{Rs. } 89{,}633.78$$

Note: The compound interest, CI = A–P = Rs.69,633.78.

Example 33: If a depositor gets Rs.30,04,166 from a bank which pays an interest of 10% p.a. compounded continously, how much did he deposit 11 years ago?

Solution: By substituting A = 30,04,166, n = 11 and i=0.1(=10/100) in the formula.

$$P = \frac{A}{e^{in}},$$

$$P = \frac{3004166}{e^{0.1 \times 11}} = \text{Rs. } 10{,}00{,}000.$$

6. Depreciation

Money lent or that deposited in a bank earns interest and accumulates time after time. But vehicles, machinery, etc. fall in value over the passage of time. The fall or decrease in value is known as depreciation. Compound interest formulae can be used in this context by taking r as the rate of depreciation and the sign of r as minus.

Example 34: At the end of each year, the value of a machine depreciates by 10% of its value at the commencement of the year. If the value of the machine at the commencement was Rs.58,750, find the value of the machine after 3 years.

(B.Com. Bharathidasan, A83)

Solution: Given: r=10; P=58,750; n=3.

∴ the value of the machine at the end of 3 years,

$$A = P\left(1 - \frac{r}{100}\right)^n = 58750\left(1 - \frac{10}{100}\right)^3$$

$$= \text{Rs. } 42,828.75$$