UNIT IV

Assignment Problem

Introduction:

The assignment problem is a particular case of transportation problem for which more efficient (less-time consuming) solution method has been devised by KUHN (1956) and FLOOD (1956). The justification of the steps leading to the solution is based on theorems proved by Hungarian Mathematicians KONEIG (1950) and EGERVARY (1953), hence the method is named Hungarian Method.

Suppose that we have 'm' jobs to be performed on 'n' machines . The cost of assigning each job to each machine is C_{ij} (i =1,2,…,n and j = 1,2,…,n).Our objective is to assign the different jobs to the different machines(one job per machine) to minimize the overall cost. This is known as **assignment problem.**

The assignment problem is a special case of transportation problem where the number of sources and destinations are equal. Supply at each source and demand at each destination must be one. It means that there is exactly one occupied cell in each row and each column of the transportation table . Jobs represent sources and machines represent destinations.

Definition and Mathematical formulation of Assignment problem

Consider the problem of assigning n jobs to n machines (one job to one machine). Let C_{ij} be the cost of assigning i^{th} job to the j^{th} machine and x_{ij} represents the assignment of i^{th} job to the j^{th} machine.

$$\text{Then, } x_{ij} = \begin{cases} 1, \text{ if } i^{th} \text{ job is assigned to } j^{th} \text{ machine} \\ 0, \text{ if } i^{th} \text{ job is not assigned to } j^{th} \text{ machine.} \end{cases}$$

machines

		Machines				
		1	2	…	n	supply
	1	(x_{11}) C_{11}	(x_{12}) C_{12}	…	(x_{1n}) C_{1n}	1
	2	(x_{21}) C_{21}	(x_{22}) C_{22}	…	(x_{2n}) C_{2n}	1
Jobs	⋮		⋮	⋮	⋮	1
	m	(x_{ij}) C_{n1}	(x_{ij}) C_{n2}	…	(x_{ij}) C_{nn}	1
	demand	b_1	b_2	…	b_n	

x_{ij} is missing in any cell means that no assignment is made between the pair of job and machine.(*i.e*) $x_{ij} = 0$.

x_{ij} presents in any cell means that an assignment is made their.In such cases $x_{ij} = 1$

Mathematical Formulation of the Assignment Problem:

Mathematically an assignment problem can be stated as follows

Minimise the total cost

$$Z = \sum_{i=i}^{n} \sum_{j=1}^{n} c_{ij} \cdot x_{ij}$$

where $x_{ij} = 1$, if i^{th} person is assgned to the j^{th} job

0, if i^{th} person is that assigned to the j^{th} job

subject to the constraints

(*i*) $\sum_{i=1}^{n} x_{ij} = 1, j = 1,2 \text{---} n$

which means that only one job is done by the i-th person, i =1,2 ---*n*

(*ii*) $\sum_{j=1}^{n} x_{ij} = 1, i = 1,2\ n$

which means that only one person should be assigned to the j^{th} job, $j = 1,2$ --- *n*

Note

The optimum assignment schedule remains unaltered if we add or subtract a constant from all the elements of the row or column of the assignment cost matrix.

Note

If for an assignment problem all $C_{ij} > 0$ then an assignment schedule (x_{ij}) which satisfies $\sum C_{ij}\, x_{ij} = 0$ must be optimal.

Unbalanced Assignment Problem:

Any assignment problem is said to be unbalanced if the cost matrix is not a square matrix, i.e. the no of rows and the no of columns are not equal. To make it balanced we add a dummy row or dummy column with all the entries as zero.

Solution of assignment problems (Hungarian Method)

First check whether the number of rows is equal to the numbers of columns, if it is so, the assignment problem is said to be balanced.

Step :1 Choose the least element in each row and subtract it from all the elements of that row.

Step :2 Choose the least element in each column and subtract it from all the elements of that column. Step 2 has to be performed from the table obtained in step 1.

Step:3 Check whether there is atleast one zero in each row and each column and make an assignment as follows.

(i) Examine the rows successively until a row with exactly one zero is found. Mark that zero by □ , that means an assignment is made there . Cross (×) all other zeros in its column. Continue this until all the rows have been examined.

(ii) Examine the columns successively until a columns with exactly one zero is found. Mark that zero by □ , that means an assignment is made there . Cross (×) all other zeros in its row. Continue this until all the columns have been examined

Step :4 If each row and each column contains exactly one assignment, then the solution is optimal.

Problem 1

Solve the following assignment problem. Cell values represent cost of assigning job A, B, C and D to the machines I, II, III and IV.

		machines			
		I	II	III	IV
jobs	*A*	10	12	19	11
	B	5	10	7	8
	C	12	14	13	11
	D	8	15	11	9

Solution:

Here the number of rows and columns are equal.

∴ The given assignment problem is balanced. Now let us find the solution.

Step 1: Select a smallest element in each row and subtract this from all the elements in its row.

	I	II	III	IV
A	0	2	9	1
B	0	5	2	3
C	1	3	2	0
D	0	7	3	1

Look for atleast one zero in each row and each column.Otherwise go to step 2.

Step 2: Select the smallest element in each column and subtract this from all the elements in its column.

	I	II	III	IV
A	0	0	7	1
B	0	3	0	3
C	1	1	0	0
D	0	5	1	1

Since each row and column contains atleast one zero, assignments can be made.

Step 3 (Assignment):

Examine the rows with exactly one zero. First three rows contain more than one zero. Go to row D. There is exactly one zero. Mark that zero by $\square$ (i.e) job D is assigned to machine I. Mark other zeros in its column by × .

	I	II	III	IV
A	$\cancel{0}$	0	7	1
B	$\cancel{0}$	3	0	3
C	1	1	0	0
D	$\boxed{0}$	5	1	2

Step 4: Now examine the columns with exactly one zero. Already there is an assignment in column I. Go to the column II. There is exactly one zero. Mark that zero by $\square$. Mark other zeros in its row by × .

$$
\begin{array}{c|cccc|}
 & \text{I} & \text{II} & \text{III} & \text{IV} \\
\hline
A & \cancel{0} & \boxed{0} & 7 & 1 \\
B & \cancel{0} & 3 & 0 & 3 \\
C & 1 & 1 & 0 & 0 \\
D & \boxed{0} & 5 & 1 & 2 \\
\hline
\end{array}
$$

Column III contains more than one zero. Therefore proceed to Column IV, there is exactly one zero. Mark that zero by $\square$. Mark other zeros in its row by × .

$$
\begin{array}{c|cccc|}
 & \text{I} & \text{II} & \text{III} & \text{IV} \\
\hline
A & \cancel{0} & \boxed{0} & 7 & 1 \\
B & \cancel{0} & 3 & 0 & 3 \\
C & 1 & 1 & \cancel{0} & \boxed{0} \\
D & \boxed{0} & 5 & 1 & 2 \\
\hline
\end{array}
$$

Step 5: Again examine the rows. Row B contains exactly one zero. Mark that zero by $\square$.

$$
\begin{array}{c|cccc|}
 & \text{I} & \text{II} & \text{III} & \text{IV} \\
\hline
A & \cancel{0} & \boxed{0} & 7 & 1 \\
B & \cancel{0} & 3 & \boxed{0} & 3 \\
C & 1 & 1 & \cancel{0} & \boxed{0} \\
D & \boxed{0} & 5 & 1 & 2 \\
\hline
\end{array}
$$

Thus all the four assignments have been made. The optimal assignment schedule and total cost is

Job	Machine	cost
A	II	12
B	III	7
C	IV	11
D	I	8
Total cost		38

The optimal assignment (minimum) cost

= Rs. 38/-

Problem 2:

Consider the problem of assigning five jobs to five persons. The assignment costs are given as follows. Determine the optimum assignment schedule.

		Job				
		1	2	3	4	5
Person	*A*	8	4	2	6	1
	B	0	9	5	5	4
	C	3	8	9	2	6
	D	4	3	1	0	3
	E	9	5	8	9	5

Solution:

Here the number of rows and columns are equal.

∴ The given assignment problem is balanced.

Now let us find the solution.

Step 1: Select a smallest element in each row and subtract this from all the elements in its row.

The cost matrix of the given assignment problem is

		Job				
		1	2	3	4	5
	A	7	3	1	5	0
Person	*B*	0	9	5	5	4
	C	1	6	7	0	4
	D	4	3	1	0	3
	E	4	0	3	4	0

Column 3 contains no zero. Go to Step 2.

Step 2: Select the smallest element in each column and subtract this from all the elements in its column.

		Job				
		1	2	3	4	5
	A	7	3	0	5	0
	B	0	9	4	5	4
Person	*C*	1	6	6	0	4
	D	4	3	0	0	3
	E	4	0	2	4	0

Since each row and column contains atleast one zero, assignments can be made.

Step 3 (Assignment):

Examine the rows with exactly one zero. Row B contains exactly one zero. Mark that zero by □ (i.e) PersonB is assigned to Job 1. Mark other zeros in its column by × .

Person \ Job	1	2	3	4	5
A	7	3	0	5	0
B	$\boxed{0}$	9	4	5	4
C	1	6	6	0	4
D	4	3	0	0	3
E	4	0	2	4	0

Now, Row C contains exactly one zero. Mark that zero by □. Mark other zeros in its column by × .

Person \ Job	1	2	3	4	5
A	7	3	0	5	0
B	$\boxed{0}$	9	4	5	4
C	1	6	6	$\boxed{0}$	4
D	4	3	0	$\times$	3
E	4	0	2	4	0

Now, Row D contains exactly one zero. Mark that zero by □. Mark other zeros in its column by × .

$$
\begin{array}{cc|ccccc|}
 & & \multicolumn{5}{c}{\text{Job}} \\
 & & 1 & 2 & 3 & 4 & 5 \\
\hline
 & A & 7 & 3 & \cancel{0} & 5 & 0 \\
 & B & \boxed{0} & 9 & 4 & 5 & 4 \\
\text{Person} & C & 1 & 6 & 6 & \boxed{0} & 4 \\
 & D & 4 & 3 & \boxed{0} & \cancel{0} & 3 \\
 & E & 4 & 0 & 2 & 4 & 0 \\
\hline
\end{array}
$$

Row E contains more than one zero, now proceed column wise. In column 1, there is an assignment. Go to column 2. There is exactly one zero. Mark that zero by □. Mark other zeros in its row by × .

$$
\begin{array}{cc|ccccc|}
 & & \multicolumn{5}{c}{\text{Job}} \\
 & & 1 & 2 & 3 & 4 & 5 \\
\hline
 & A & 7 & 3 & \cancel{0} & 5 & 0 \\
 & B & \boxed{0} & 9 & 4 & 5 & 4 \\
\text{Person} & C & 1 & 6 & 6 & \boxed{0} & 4 \\
 & D & 4 & 3 & \boxed{0} & \cancel{0} & 3 \\
 & E & 4 & \boxed{0} & 2 & 4 & \cancel{0} \\
\hline
\end{array}
$$

There is an assignment in Column 3 and column 4. Go to Column 5. There is exactly one zero. Mark that zero by □. Mark other zeros in its row by × .

$$
\begin{array}{cc|ccccc|}
 & & & & \text{Job} & & \\
 & & 1 & 2 & 3 & 4 & 5 \\
\hline
 & A & 7 & 3 & \cancel{0} & 5 & \boxed{0} \\
 & B & \boxed{0} & 9 & 4 & 5 & 4 \\
\text{Person} & C & 1 & 6 & 6 & \boxed{0} & 4 \\
 & D & 4 & 3 & \boxed{0} & \cancel{0} & 3 \\
 & E & 4 & \boxed{0} & 2 & 4 & \cancel{0} \\
\hline
\end{array}
$$

Thus all the five assignments have been made. The Optimal assignment schedule and total cost is

Person	Job	cost
A	5	1
B	1	0
C	4	2
D	3	1
E	2	5
Total cost		9

The optimal assignment (minimum) cost = Rs. 9/-

Problem 3:

Solve the following assignment problem.

		Men		
		1	2	3
Task	P	9	26	15
	Q	13	27	6
	R	35	20	15
	S	18	30	20

Solution:

Since the number of columns is less than the number of rows, given assignment problem is unbalanced one. To balance it , introduce a dummy column with all the entries zero. The revised assignment problem is

		Men			
		1	2	3	d
Task	P	9	26	15	0
	Q	13	27	6	0
	R	35	20	15	0
	S	18	30	20	0

Here only 3 tasks can be assigned to 3 men.

Step 1: is not necessary, since each row contains zero entry. Go to Step 2.

Step 2 :

		Men			
		1	2	3	d
Task	P	0	6	9	0
	Q	4	7	0	0
	R	26	0	9	0
	S	9	10	14	0

Step 3 (Assignment) :

$$
\begin{array}{cc|cccc|}
 & & & & \text{Men} & \\
 & & 1 & 2 & 3 & d \\
\hline
 & P & \boxed{0} & 6 & 9 & \cancel{0} \\
\text{Task} & Q & 4 & 7 & \boxed{0} & \cancel{0} \\
 & R & 26 & \boxed{0} & 9 & \cancel{0} \\
 & S & 9 & 10 & 14 & \boxed{0} \\
\hline
\end{array}
$$

Since each row and each column contains exactly one assignment, all the three men have been assigned a task. But task S is not assigned to any Man. The optimal assignment schedule and total cost is

Task	Men	cost
P	1	9
Q	3	6
R	2	20
S	*d*	0
Total cost		35

The optimal assignment (minimum) cost = Rs.35/-

Travelling salesman problem

A salesman normally must visit a number of cities starting from his head quarters. The distance between every pair of cities are assumed to be known. The problem of finding the shortest distance if the salesman starts from his head quarters and passes through each city exactly once and returns to the headquarters is called Traveling Salesman problem.

Problem 4:

Find Solution of Travelling salesman problem (MIN case)

Work\Job	A	B	C	D	E
A	x	2	5	7	1
B	6	x	3	8	2
C	8	7	x	4	7

D	12	4	6	x	5
E	1	3	2	8	x

Solution:

The number of rows = 5 and columns = 5

	A	*B*	*C*	*D*	*E*
A	M	2	5	7	1
B	6	M	3	8	2
C	8	7	M	4	7
D	12	4	6	M	5
E	1	3	2	8	M

Step-1: Find out the each row minimum element and subtract it from that row

	A	*B*	*C*	*D*	*E*
A	M	1	4	6	0
B	4	M	1	6	0
C	4	3	M	0	3
D	8	0	2	M	1
E	0	2	1	7	M

Step-2: Find out the each column minimum element and subtract it from that column.

	A	*B*	*C*	*D*	*E*
A	M	1	3	6	0
B	4	M	0	6	0
C	4	3	M	0	3
D	8	0	1	M	1
E	0	2	0	7	M

Rowwise & columnwise assignment shown in table

	A	*B*	*C*	*D*	*E*
A	M	1	3	6	**[0]**
B	4	M	**[0]**	6	0
C	4	3	M	**[0]**	3
D	8	**[0]**	1	M	1
E	**[0]**	2	0	7	M

Step-4: Number of assignments = 5, number of rows = 5
The solution gives the sequence : A→E,E→A
The above solution is not a solution to the travelling salesman problem as he visits each city only once.The next best solution can be obtained by bringing the minimum non-zero element, i.e., 1 into the solution.The cost 1 occurs at 3 places. We will consider all the cases separately until the acceptable solution is obtained.Case: 1 of 3 for minimum non-zero element 1

	A	*B*	*C*	*D*	*E*
A	M	**[1]**	3	6	0
B	4	M	**[0]**	6	0
C	4	3	M	**[0]**	3
D	8	0	1	M	**[1]**
E	**[0]**	2	0	7	M

Step-4: Number of assignments = 5, number of rows = 5
The solution gives the sequence : A→B,B→C,C→D,D→E,E→A
So Optimal assignments are

	A	*B*	*C*	*D*	*E*	
A	M	**[1]**	3	6	0	
B	4	M	**[0]**	6	0	
C	4	3	M	**[0]**	3	
D	8	0	1	M	**[1]**	
E	**[0]**	2	0	7	M	

Optimal solution is

Work	Job	Cost
A	*B*	2
B	*C*	3
C	*D*	4
D	*E*	5
E	*A*	1
	Total	15