Unit II
Simplex Method of Linear Programming

Under 'Graphical solutions' to LP, the objective function obviously should have not more than two decision variables. If the decision variables are more than two, the 'Cartesian Plane' cannot accommodate them. And hence, a most popular and widely used analysis called 'SIMPLEX METHOD', is used. This method of analysis was developed by one American Mathematician by name George B. Dantzig, during 1947.

This method provides an algorithm (a procedure which is iterative) which is based on fundamental theorems of Linear Programming. It helps in moving from one basic feasible solution to another in a prescribed manner such that the value of the objective function is improved. This procedure of jumping from one vertex to another vertex is repeated.

1.Solve the following LPP by simplex method

Maximize 'Z' = $2x_1 + 3x_2$

$x_1 + x_2 \leq 1$

$3x_1 + x_2 \leq 4$

Where, $x_1, x_2 \geq 0$

Solution

Step 1: Converting the inequalities into equalities by adding slack variables and the standard form of LPP is given as.

Maximize z = $2X_1 + 3x_2 + 0S_1 + 0\,S_2$

$x_1 + x_2 + s_1 + 0\,S_2 = 1$

$3x_1 + x_2 + 0s_1 + s_2 = 4$

and $x_1, x_2 \geq 0$

Step 2: First iteration of Simplex Method.

Objective function C_j			2	3	0	0	Min. Ratio (Positive) X_B/a_{ir}
BV	C_B	X_B	x_1	x_2	S_1	S_2	
S_1	0	1	1	1	1	0	1/1 = 1 (KR) →S_1leaves
S_2	0	4	3	1	0	1	4/1 = 4
Z_j		0	0	0	0	0	
$Z_j - C_j$		0	-2	-3	0	0	

↑ X_2 Enters (KC)

X_2 Enters and S_1 leaves

Since there are some $Z_j - C_j < 0$, the current basic feasible solution is not optimal

Step 3: Second iteration of Simplex Method.

Objective function C_j			2	3	0	0	Min. Ratio (Positive) X_B/a_{ir}
BV	C_B	X_B	x_1	x_2	S_1	S_2	
x_2	3	1/1 = 1	1/1 = 1	1/1 = 1	1/1 = 1	0/1=0	--
S_2	0	4-1= 3	3-1 = 2	1-1= 0	0-1 =-1	1-1 =0	--
Z_j (CX)		3	3	3	3	0	
$Z_j - C_j$		3	1	0	3	0	

Since all $Z_j - C_j \geq 0$, the current basic feasible solution is optimal

Therefore the Optimal Solution is

Maximize Z = 3

$X_1 = 0$

$X_2 = 1$

<u>Verification:</u>

Maximize z = $\mathbf{2X_1 + 3x_2}$

$= 2(0) + 3(1) = 3$

2. Solve the following LPP by simplex method:

Maximise 'Z' = $5x_1 + 3x_2$ [Subject to constraints]

$$x_1 + x_2 \leq 2$$

$$5x_1 + 2x_2 \leq 10$$

$$3x_1 + 8x_2 \leq 12$$

Where, $x_1, x_2 \geq 0$ [Non-negativity constraints]

Solution:

Step 1: Conversion of inequalities into equalities adding slack variables

$$x_1 + x_2 + s_1 = 2$$

$$5x_1 + 2x_2 + s_2 = 10$$

$$3x_1 + 8x_2 + s_3 = 12$$

Where, x_3, x_4 and x_5 are slack variables.

Step 2: Fit the data into first iteration of Simplex Method

BV	C_B	X_B	x_1	x_2	S_1	S_2	S_3	Min. Ratio
S_1	0	2	1	1	1	0	0	2/1 = 2(KR) → S_1 leaves
S_2	0	10	5	2	0	1	0	10/5 = 2
S_3	0	12	3	8	0	0	1	12/3 = 4
Z_j		10	0	0	0	0	0	
C_j		10	5	3	0	0	0	
$Z_j – C_j$		10	-5	-3	0	0	0	

(↑ KC) X$_1$ Enters and S$_1$ leaves

Since there are some $Z_j - C_j <$ 0, the current basic feasible solution is not optimal

Step 3: Fit the data into second iteration of Simplex Method.

BV	C_B	X_B	x_1	x_2	S_1	S_2	S_3	Min. Ratio
x_1	5	2/1 = 2	1/1 = 1	1/1 = 1	1/1 = 1	0	0	–
S_2	0	10 – 2(5) = 0	5 – 1(5) = 0	2–1(5) = –3	0	1-0(5)=1	0	–
S_3	0	12 – 2(3) = 6	3 – 1(3) = 0	8–1(3) = 5	0	0	1-0(3)=1	–
	Z_j	10	5	5	1	0	0	
	C_j	10	5	3	0	0	0	
	$Z_j - C_j$	10	0	2	1	0	0	

Since all $Z_j - C_j \geq$ 0, the current basic feasible solution is optimal

Therefore the Optimal Solution is

Maximize Z = 10

$X_1 = 2$

$X_2 = 0$

Verification:

Maximize z = $5x_1 + 3x_2$ = 5(2) + 3(0) =10

3.Solve the following LPP by simplex method

Minimise 'Z' = $x_1 - 3x_2 + 2x_3$

Subject to constraints $3x_1 - x_2 + 3x_3 \leq 7$

$-2x_1 + 4x_2 \leq 12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

Where, $x_1, x_2, x_3 \geq 0$ [Non-negativity constraints]

Solution: ***Step 1:*** Conversion of the minimization case into maximisation case.

Maximise Z = $-x_1 + 3x_2 - 2x_3$

Step 2: Convert of the inequalities into equalities by adding slack variables.

$$3x_1 - x_2 + 3x_3 + S_1 = 7$$

$$-2x_1 + 4x_2 + S_2 = 12$$

$$-4x_1 + 3x_2 + 8x_3 + S_3 = 10$$

Step 3: First iteration of Simplex Method.

C_j			-1	3	-2	0	0	0	Min. Ratio of positive X_B/a_{ir}
C_B	Y_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	
0	S_1	7	3	-1	3	1	0	0	-
0	S_2	12	-2	4	0	0	1	0	$12/4 = 3 \rightarrow$
0	S_3	10	-4	3	8	0	0	1	$10/3 = 3.33$
Z_j		0	0	0	0	0	0	0	
$Z_j - C_j$		0	1	$-3 \uparrow$	2	0	0	0	

X_1 Enters and S_2 leaves

Since there are some $Z_j - C_j < 0$, the current basic feasible solution is not optimal

Rough work for Table 2

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	X_B	X_1	X_2	X_3	S_1	S_2	S_3
Old S_2 Equation	12	-2	4	0	0	1	0
	/4	/4	/4	/4	/4	/4	/4
New X_2 Equation	3	-1/2	1	0	0	1/4	0

	X_B	X_1	X_2	X_3	S_1	S_2	S_3
Old S_1 Equation	7	3	-1	3	1	0	0
New X_2 Equation	3	-1/2	1	0	0	1/4	0
New S_1 Equation	10	5/2	0	3	1	1/4	0

	X_B	X_1	X_2	X_3	S_1	S_2	S_3
Old S_3 Equation	10	−4	3	8	0	0	1
New X_2 Equation	3(-3)	-1/2(-3)	1(-3)	0(-3)	0(-3)	1/4(-3)	0(-3)
New S_3 Equation	1	-5/2	0	8	0	-3/4	1

Step 4: Table 2 Second iteration

C_j			-1	3	-2	0	0	0	Min. Ratio of positive X_B/a_{ir}
C_B	Y_B	X_B	X_1	X_2	Y_3	S_1	S_2	S_3	
0	S_1	10	5/2	0	3	1	1/4	0	10/2.5 = 4 s_1 leaves
3	X_2	3	-1/2	1	0	0	1/4	0	- (negative)
0	S_3	1	-5/2	0	8	0	-3/4	1	- (negative)
	Z_j	9	-3/2	3	0	0	3/4	0	
$Z_j - C_j$		9	-1/2	0	2	0	3/4	0	

X_1 Enters

X_1 Enters and $S1$ leaves Since there are some $Z_j - C_j <$ 0, the current basic feasible solution is not optimal

Rough work for Table 3

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	X_B	X_1	X_2	X_3	S_1	S_2	S_3
Old S_1 Equation	10	5/2	0	3	1	1/4	0
	X 2/5	X 2/5	X 2/5	X 2/5	X 2/5	X 2/5	X 2/5
New X_1 Equation	4	1	0	6/5	2/5	1/10	0

	X_B	X_1	X_2	X_3	S_1	S_2	S_3
Old X_2 Equation	3	-1/2	1	0	0	1/4	0
New X_1 Equation	+4(1/2)	+1(1/2)	+0(1/2)	+6/5(1/2)	+2/5(1/2)	+1/10(1/2)	+0(1/2)

New X_2Equation	5	0	1	3/5	1/5	3/10	0

	X_B	X_1	X_2	X_3	S_1	S_2	S_3
Old S_3 Equation	1	-5/2	0	8	0	-3/4	1
New X_1 Equation	+4(5/2)	+1(5/2)	+0(5/2)	+6/5(5/2)	+2/5(5/2)	+1/10(5/2)	+0(5/2)
New S_3 Equation	11	0	0	3	1	-1/2	1

Step 5: Third iteration .

C_j			-1	3	-2	0	0	0
C_B	Y_B	X_B	X_1	X_2	x_3	S_1	S_2	S_3
-1	X_1	4	1	0	6/5	2/5	1/10	0
3	X_2	5	0	1	3/5	1/5	3/10	0
0	S_3	11	0	0	11	1	-1/2	1
Z_j		11	-1	3	3/5	1/5	8/10	0
$Z_j - C_j$		11	0	0	13/5	1/5	8/10	0

Since all $Z_j - C_j \geq 0$, the current basic feasible solution is optimal

Therefore, Maximise 'Z' $= -x_1 + 3x_2 - 2x_3$

$= -4 + 3(5) - 2(0)$

$= 11$

Therefore, Minimize 'Z' $= x_1 - 3x_2 + 2x_3$

$= 4 - 3(5) + 2(0)$

Minimize

$Z = -11$

$X_1 = 4$

$X_2 = 5$

<u>Big 'M' Method</u>

When the Linear Programming problem has greater than (≥) or equal to (=)types of equations as constraints, it is obvious that some quantity should be deducted to convert them into equalities. The variables attached to it are known as 'surplus variables'. If the inequality is of the type greater than or equal to then add surplus variables which carry a negative sign and their cost coefficients in the objective function would be zero. As they would be considering slack or artificial variables initially as basic variables, and as the surplus variables carry negative signs, which cannot be taken into the basis. Hence, artificial variables along with the surplus variables would be added.

These artificial variables carry large negative values (-M) in the objective function. The artificial variables can also be added to the equation. This helps in choosing the initial variable or variables for the basis. The slack variables then would go to the basis whose cost coefficients are supposed to be zero (0) and the cost coefficients of artificial variables are supposed to be -M for maximization cases and +M for Minimization cases. The procedure for iteration follows when Simplex technique to obtain the optimum solution is used. Since the method involves artificial variables carrying -M as the cost coefficient, where M is a very large number which helps in the optimum solution finding and hence it is known as **'Big M Method'**.

Steps:

1. Express the problem in the standard form by using slack, surplus and artificial variables.
2. Select slack variables and artificial variables as the initial basic variables with the cost coefficients as '0' or '-M' respectively.
3. Use simplex procedure for iterations & obtain optimum solution. During the iterations, one can notice that the artificial variables leave the basis first and then the slack variables with improved value of objective function at each iteration to obtain the optimum solution.

4. Use Big M method to solve the following

Maximize $Z = 2X_1 + X_2 + X_3$

Subject to constraints $4X_1 + 6X_2 + 3X_3 \leq 8$

$3X_1 - 6X_2 - 4X_3 \leq 1$

$2X_1 + 3X_2 - 5X_3 \geq 4$

and $X_1, X_2, X_3 \geq 0$

Solution:

Step 1: Convert the inequalities into equalities by adding slack, surplus and artificial variables.

Maximize $Z = 2X_1 + X_2 + X_3 + 0S_1 + 0S_2 + 0S_3 - MR1$

Subject to constraints $4X_1 + 6X_2 + 3X_3 + S_1 + 0S_2 + 0S_3 = 8$

$3X_1 - 6X_2 - 4X_3 + 0S_1 + S_2 + 0S_3 = 1$

$2X_1 + 3X_2 - 5X_3 + 0S_1 + 0S_2 - S_3 + R_1 = 4$

And $X_1, X_2, X_3, S_1, S_2, S_3, R_1 \geq 0$

Step 2: Table 1 First iteration of Simplex Method.

C_j			2	1	1	0	0	0	-M	Min. Ratio of positive X_B/a_{ir}
C_B	Y_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	R1	
0	S_1	8	4	6	3	1	0	0	0	8/6 =1.3
0	S_2	1	3	-6	-4	0	1	0	0	-
-M	R_1	4	2	3	-5	0	0	-1	1	$4/3 = 1.33 \rightarrow R_1$
Z_j		-4M	-2M	-3M	5M	0	0	M	-M	
$Z_j - C_j$		-4M	-2M-2	-3M-1	5M-1	0	0	M	0	
				$\uparrow X_2$						

X_2 Enters and R_1 leaves

Since there are some $Z_j - C_j < 0$, the current basic feasible solution is not optimal

Rough work for Table 2

R1	4	2	3	-5	0	0	-1

	/3	/3	/3	/3	/3	/3	/3
	4/3	2/3	1	-5/3	0	0	-1/3

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S_1	8	4	6	3	1	0	0
	4/3(-6)	2/3(-6)	1(-6)	-5/3 (-6)	0(-6)	0(-6)	-1/3(-6)
	0	0	0	13	1	0	2

S_2	1	3	-6	-4	0	1	0
	4/3(6)	2/3(6)	1(6)	- 5/3(6)	0(6)	0(6)	-1/3(6)
	9	7	0	-14	0	1	-2

C_j			2	1	**1**	**0**	**0**	**0**	Min. Ratio of positive X_B/a_{ir}
C_B	**Y_B**	**X_B**	**X_1**	**X_2**	**X_3**	**S_1**	**S_2**	**S_3**	
0	S_1	0	0	0	13	1	0	2	$\rightarrow S_1$
0	S_2	9	7	0	-14	0	1	-2	-
1	X_2	4/3	2/3	1	-5/3	0	0	-1/3	-
Z_j		4/3	2/3	1	-5/3	0	0	-1/3	
$Z_j - C_j$		4/3	-4/3	0	-8/3	0	0	-1/3	
					$\uparrow X_3$				

Step 3: Table 2

second iteration of Simplex Method.

X_3 Enters and S_1 leaves

Since there are some $Z_j - C_j < 0$, the current basic feasible solution is not optimal

Step 4: Table 3 Third iteration of Simplex Method.

C_j			2	1	1	0	0	0	Min. Ratio of positive X_B/a_{ir}
C_B	Y_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	
1	X_3	0	0	0	1	1/13	0	2/13	-
0	S_2	9	7	0	0	14/13	1	2/13	9/7=1.2 →$s_2$leaves
1	X_2	4/3	2/3	1	0	5/39	0	-1/13	2
Z_j		4/3	2/3	1	1	8/39	0	1/13	
$Z_j - C_j$		4/3	-4/3	0	0	8/39	0	1/13	
			X_1 ↑ enters						

X_1 Enters and S_2 leaves

Since there are some $Z_j - C_j <$ 0, the current basic feasible solution is not optimal

Step 5: Table 4 Fourth iteration of Simplex Method.

C_j			2	1	1	0	0	0
C_B	Y_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3
1	X_3	0	0	0	1	1/13	0	2/13
2	X_1	9/7	1	0	0	2/13	1/7	2/91
1	X_2	10/21	0	1	0	1/39	-2/21	-25/273
Z_j		64/21	2	1	1	16/39	4/21	29/273
$Z_j - C_j$		64/21	0	0	0	16/39	4/21	29/273

Since all $Zj - Cj \geq 0$, the current basic feasible solution is optimal

Therefore the Optimal Solution is

Maximize $Z = 64/21$
$X_1 = 9/7$
$X_2 = 10/21$
$X_3 = 0$

Duality in LPP – Formulation of Dual LPP (concept only) –Dual primal relationship.

For every linear programming problem, there is an associated linear programming problem. The former problem is called primal and the latter is called its dual and vice versa. The two problems may appear to have superficial relationship between each other but they possess very intimately related properties and useful one, so that the optimal solution of one problem gives complete information about the optimal solution to the other. In other words, the optimal solutions for both the problems are same.

The concept of duality is very much useful to obtain additional information about the variations in the optimal solutions when certain changes are affected in the constraint coefficients, resource availabilities and objective function coefficients. This is termed as post-optimality or sensitivity analysis.

Concept of Duality

One part of a Linear Programming Problem (LPP) is called the *Primal* and the other part is called the *Dual*. In other words, each maximization problem in LP has its corresponding problem, called the dual, which is a minimization problem. Similarly, each minimization problem has its corresponding dual, a maximization problem. For example, if the primal is concerned with maximizing the contribution from the three products A, B, and C and from the three departments X, Y, and Z, then the dual will be concerned with minimizing the costs associated with the time used in the three departments to produce those three products. An optimal solution from the primal and the dual problem would be same as they both originate from the same set of data.

Definition

The **Duality in Linear Programming** states that every linear programming problem has another linear programming problem related to it and thus can be derived from it. The original linear programming problem is called **"Primal,"** while the derived linear problem is called **"Dual."**

Formulation of Dual LPP

The following procedure is adopted to convert primal problem into its dual. Simplex method is applied to obtain the optimal solution for both primal and dual form.

Step 1: For each constraint, in primal problem there is an associated variable in the dual problem.

Step 2: The elements of right-hand side of the constraints will be taken as the coefficients of the objective function in the dual problem.

Step 3: If the primal problem is maximization, then its dual problem will be minimization and vice versa.

Step 4: The inequalities of the constraints should be interchanged from to ≤ and vice versa and the variables in both the problems are non-negative.

Step 5: The rows of primal problem are changed to columns in the dual problem

Step 6: The coefficients of the objective function will be taken to the right hand side of the constraints of the dual problem.

Advantages of Duality

1. It is advantageous to solve the dual of a primal having less number of constraints because the number of constraints usually equals the number of iterations required to solve the problem.
2. It avoids the necessity for adding surplus or artificial variables and solves the problem quickly (the technique is known as the primal-dual method).
3. The dual variables provide an important economic interpretation of the final solution of an LP problem.
4. It is quite useful when investigating changes in the parameters of an LPP (the technique known as sensitivity analysis).
5. Duality is used to solve an LPP by the simplex method in which the initial solution is infeasible.

Primal Dual Relationship

- The number of constraints in the primal problem is equal to the number of dual variables, and *vice versa*.
- If the primal problem is a maximization problem, then the dual problem is a minimization problem and *vice versa*.
- If the primal problem has greater than or equal to type constraints, then the dual problem has less than or equal to type constraints and *vice versa*.
- The profit coefficients of the primal problem appear on the right-hand side of the dual problem.
- The rows in the primal become columns in the dual and *vice versa*.

  All primal and dual variables must be non-negative (>0).

Problems in Dual Formulation of Linear Programming

Example 1: Write the dual of the following

problem. Maximise 'Z' = $-6x_1 + 7x_2$

Subject to, $-x_1 + 2x_2 \leq -5$

$3x_1 + 4x_2 \leq 7$

$x_1, x_2 \geq 0$

Minimise $Z = -5y_1 + 7y_2$

$-y_1 + 3y_2 \geq -6$

$2y_1 + 4y_2 \geq 7$

$y_1, y_2 \geq 0$

Solution: The given problem is considered as primal linear programming problem. To convert it into dual, the following procedure is adopted.

Step 1: There are 2 constraints and hence the dual problem will have 2 variables. Let us denote them as y_1 and y_2.

Step 2: The right hand side of the constraints are -5 and 7 which are taken as the coefficients of the variables y_1 and y_2 in the objective function.

Step 3: The primal objective problem is maximization and hence the dual seeks minimization for the objective function. Hence, the objective functions for the dual problem is given by

$$\text{Minimise } Z = -5y_1 + 7y_2$$

Step 4: The inequalities of the constraints for the primal problem are of the type ($\leq$). Hence, the inequalities for the dual constraints will be of the type ($\geq$).

Step 5: The rows of primal problem are changed to columns in the dual problem

Step 6: The coefficients of the objective function for the given primal are -6 and 7. They are taken on the right hand side of the constraints for the dual problem. Hence, the constraints for the dual problem are represented as

$$-y_1 + 3y_2 \geq -6$$
$$2y_1 + 4y_2 \geq 7$$
$$y_1, y_2 \geq 0$$

Example 2: Find the dual of the following problem:

Minimize $z = 3x_1 + 3x_2$

subject to

$2x_1 + 4x_2 \geq 40$
$3x_1 + 2x_2 \geq 50$

$x_1, x_2 \geq 0$

Solution.

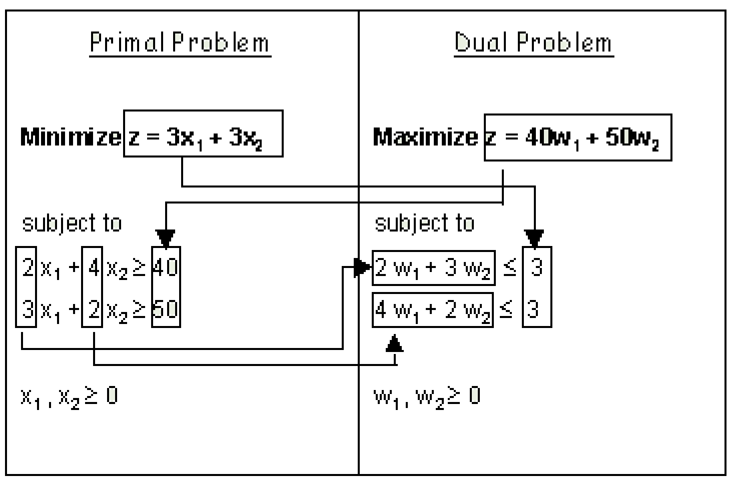
Maximize $z = 40w_1 + 50w_2$

subject to

$2w_1 + 3w_2 \leq 3$
$4w_1 + 2w_2 \leq 3$

$w_1, w_2 \geq 0$

Primal Dual Relationship in Linear Programming (LP)



Example 3: Find the dual of the following problem:
Maximize$Z = 50x_1+30x_2$
Subject to:
$4x_1 + 3x_2 \leq 100$
$3x_1 + 5x_2 \leq 150$
$X_1, x_2 \geq 0$
The duality can be applied to the above original linear programming problem as:
Minimize
$G = 100y_1+150y_2$
Subject to:
$4y_1 + 3y_2 \geq 50$
$3y_1 +5y_2 \geq 30$

$Y_1, y_2 \geq 0$