Design of experiments

UNIT V

FACTORIAL EXPERIMENTS

In Randomised Block Design (RBD) or in Latin Square Design (LSD) the testing of a number of treatments not necessarily related were tested. In industrial applications frequently that several factors may affect the characteristic in which we are interested, and wish to estimate the effects of each of the factors and how the effect of one factor varies over the level of other factors.

For example, The yield of a chemical process may be affected by several factors such as the levels of pressure, temperature, rate of agitation, and proportions of reactants etc. One might try to test each of the factors separately, holding all other factors constant in a given experiment. The logical procedure would be to vary all factors simultaneously, i.e., within framework of the same experiment. This is called as factorial experiment.

The factorial experiments are particularly useful in experimental situations which require the examination of the effects of varying two or more factors.

In CRD, RBD and LSD the comparison and estimation of the effects of a single set of treatments like varieties of wheat, manure, of different methods of cultivation etc,. such experiments which deal with only one factor only may be called simple experiments.

In factorial experiment the effect of several factors of variation are studied and investigated simultaneously, the treatments being all the combinations of different factors under study. In these experiments an attempt is made to estimate the effects of each of the factors and also interaction effects i.e., the variation in the effect of one factor as a result to different levels of other factors.

As a simple illustration let us consider two fertilizers, say, Potash (K) and Nitrogen (N). Let us suppose that there are p different varieties of Potash and q different varieties of Nitrogen. P and q are termed as the levels of factors Potash and Nitrogen respectively. To find the effectiveness of various treatments, viz the different levels of Potash or Nitrogen we might conduct two simple experiments one for Potash and one for Nitrogen. A series of experiments in which only one factor is varied at a time would be both lengthy and costly. These simple experiments do not give us any information regarding the dependence or independence of one factor on the other, i.e., they do not tell us about the interaction effect (NK). The only alternative is try to investigate the variations in several factors simultaneously by conducting the above experiment as p x q factorial experiment, where p and q are the levels of various factors under consideration. In general, if the levels of various factors are qual then s^n factorial experiment means an experiment with n factors each at s levels where n is any positive integer greater than or equal to 2, for example2^3 experiment means an experiment with 3 factors each at 2 levels and 3^2 experiment means an experiment with 2 factors at 3 levels each.

ADVANTAGES OF FACTORIAL EXPERIMENT

1. It increases the scope of the experiment and its inductive value by giving information not only on the main factors but on their interactions.
2. The various levels of one factor constitute replication of other factors and increase the amount of information obtained on all factors.
3. When there are no interactions, the factorial design gives the maximum efficiency in the estimates of the effects.
4. When interactions exist, their nature being unknown a factorial design is necessary to avoid misleading conclusions.
5. In the factorial design the effect of a factor is estimated at different levels of other factors and the conclusions hold over a wide range of conditions.

2^2 FACTORIAL DESIGN

Here we have two factors each at two levels (0,1), say, so that there are 2 x 2 = 4 treatment combinations in all. Following the notations due to Yates let the capital letters A and B indicate the names of two factors under study and let the small letters a and b denote one of the two levels of each of the corresponding factors and this will be called the second level. The first level of A and B is generally expressed by the absence of the corresponding letters in the treatment combinations, the four treatment combinations can be given as follows

a_0b_0 or '1' : factors A and B both at first level

a_1b_0 or a : factor A at second level and B at first level

a_0b_1 or b : factor A at first level and B at first level

a_1b_1 or ab : factors A and B both at second level

Main effects and interaction effects

Suppose the factorial experiment with 2^2 = 4 treatments is conducted in r blocks or replicates. Let [1],[a],[b],and [ab] denote the total yields of the r units(plots)receiving the treatments 1, a, b, and ab respectively and let the corresponding mean values obtained on dividing these totals by r be denoted by (1), (a), (b) and (ab) respectively.

The effect of factor A at the first level b_0 of B = $(a_1b_0) – (a_0b_0) = (a) – (1)$ ……..(I)

Similarly,

the effect of A at the second level b_1 0f B = $(a_1b_1) – (a_0b_1) = (ab) – (b)$ …….(II)

These two effects given in (I) and (II) are called simple effects of factor A.

The average observed effect of A over the two levels of B is called the **main effect due to A** and is defined by $A = ½ [(ab) – (b) + (a) – (1)]$

$$A = ½ [b(a - 1) + (a–1)] =1/2 [(a – 1) (b + 1)]$$

Where the right-hand side is to be expanded algebraically and then the treatment combinations are to be replaced by treatment means.

similarly **the main effects of factor B is**

B = ½ [(ab) – (a) +(b) – (1)] or

B = ½ [(a + 1) (b – 1)]

Where the right-hand side is to be expanded algebraically and then the treatment combinations are to be replaced by treatment means.

The interaction of two factors is the failure of the levels of one factor, say, A to retain the same order and magnitude of performance throughout all levels of the second factor, say, B. if the two factors act independently of one another, the true effect of one would be the same at either level of other. If the two factors are not independent, the two expressions in (I) and (II) will not be the same and the difference of these two numbers is a measure of the extent to which the factors interact and **the two-factor interaction or the first order interaction** between the factors A and B as:

AB = ½ [(ab) – (b) – (a) + (1)] or

AB = ½ [(a – 1) (b – 1)]

Where the right-hand side is to be expanded algebraically and then the treatment combinations are to be replaced by treatment means.

CONTRAST

Definition: A linear combination $\sum_{i=1}^{k} c_i t_i$ of k treatment means t_i (i = 1,2, … ,k) is called a **contrast(or a comparison)** of treatment means if $\sum_{i=1}^{k} c_i = 0$. In other words, contrast is a linear combination of treatment means such that the sum of the co-efficients is zero.

ORTHOGONAL CONTRASTS

Definition: Two contrasts of k-treatment means t_i, (I =1,2, … ,k)

$$\sum_{i=1}^{k} c_i t_i\, ,\ \sum_{i=1}^{k} c_i = 0.$$

and $$\sum_{i=1}^{k} d_i t_i\, ,\ \sum_{i=1}^{k} d_i = 0.$$

are said to be orthogonal if $\sum_{i=1}^{k} c_i d_i = 0$.

Let M denote the mean yield of the four treatment combinations. Then

M = ¼ [(ab) + (a) + (b) + (1)] = ¼ [(a+1)(b+1)]

The general mean M and three mutually orthogonal contrasts viz., the main effects A and B and the interaction AB are summarized in terms of treatment combinations in the following table.

The signs for the mean effects are written according to the following rule. " give a plus sign to each of the treatments means whenever the corresponding factor is at the second level, otherwise a negative sign. For a two-factor interaction, the signs to be attached to various treatment means are obtained by multiplying the signs of the corresponding main effects.

TABLE OF SIGNS AND DIVISORS GIVING M, THE MAIN EFFECTS AND INTERACTIONS IN TERMS OF INDIVIDUAL TREATMENT MEANS

Factorial experiments	Treatments				Divisor
	(1)	(a)	(b)	(ab)	
M	+	+	+	+	4
A	-	+	-	+	2
B	-	-	+	+	2
AB	+	-	-	+	2

YATES METHOD OF COMPUTING FACTORIAL EFFECT TOTALS

For calculation of various effect totals for 2^n factorial experiment F.Yates developed a special computational rule. Yates' method consists of the following steps:

1. In the first column we write the treatment combinations. It is an essential that the treatment combinations should be written in a systematic order as explained below:
   " starting with the treatment combination 1, each factor is introduced in turn and then it is followed by all the combinations of itself with the treatment combinations previously written. For example, for 2^2 experiment with factors A and B the order of treatment combinations will be 1, a, b, ab and for 2^3 factorial experiment with factors A,B and C, the order of treatment combinations will be 1, a, b, ab, c, ac, bc, abc and so on.
2. Against each treatment combination, write the corresponding total yields from all the replicates.
3. The entries in the third column can be split into two halves. The first half is obtained by writing down in order, the pairwise sums of the values in column 2 and the second half is obtained by writing in the same order the pairwise differences of the values in the second column. The first member is to be subtracted from the second member of a pair.

4. To complete the next (4th) column, the whole of the whole of the procedure as explained in step 3 is repeated on column 3, and for 2^3 design, the 5th column is derived from 4th in a similar manner.

Thus for a 2^n factorial experiment there will be n cycles of this ‘sum and difference’ procedure. The first term in the last, viz., (n+2)th column always gives the grand total (G) while the other entries in the last column are the totals of the main effects or the interactions corresponding to the treatment combinations in the first column of the table.

YATES’METHOD FOR A 2^2 EXPERIMENT

Treatment Combination (1)	Total yield from all replicates (2)	(3)	(4)	Effect totals
‘1’	[1]	[1]+[a]	[1]+[a]+ [b]+[ab]	Grand total
a	[a]	[b]+[ab]	[ab] – [b] + [a] – [1]	[A]
b	[b]	[a] – [1]	[ab] + [b] - [a] – [1]	[B]
ab	[ab]	[ab] – [b]	[ab] – [b] - [a] + [1]	[AB]

YATES’METHOD FOR A 2^3 EXPERIMENT

Treatment Combination (1)	Total yield from all replicates (2)	(3)	(4)		Effect totals
‘1’	[1]	[1]+[a]= u_1(say)	$u_1+u_2=v_1$	$v_1+v_2=w_1$	Grand total
a	[a]	[b]+[ab]= u_2(say)	$u_3+u_4=v_2$	$v_3+v_4=w_2$	[A]
b	[b]	[c] + [ac]= u_3(say)	$u_5+u_6=v_3$	$v_5+v_6=w_3$	[B]
ab	[ab]	[bc] + [abc]= u_4(say)	$u_7+u_8=v_4$	$v_7+v_8=w_4$	[AB]
c	[c]	[a] - [1]= u_5(say)	$u_2 - u_1=v_5$	$v_2 - v_1=w_5$	[C]
ac	[ac]	[ab]-[b]= u_6(say)	$u_4 - u_3=v_6$	$v_4 - v_3=w_6$	[AC]
bc	[bc]	[ac] - [c]= u_7(say)	$u_6 - u_5=v_7$	$v_6 - v_5=w_7$	[BC]
abc	[abc]	[abc] - [bc]= u_8(say)	$u_8 - u_7=v_8$	$v_8 - v_7=w_8$	[ABC]

STATISTICAL ANALYSIS OF 2^2-DESIGN.

Factorial experiments are conducted either using CRD or RBD or LSD and thus they can be analyzed in the usual manner except that in this case treatments S.S. is split into three orthogonal components each with 1 d.f. . The main effects A and B and the interaction AB are mutually orthogonal contrasts of the treatment means.

Null Hypothesis:

H_0 : the blocks as well as treatments are homogeneous.
H_1 : the blocks as well as treatments are not homogeneous.

Total S.Sq. = $\sum_i \sum_j (y_{ij} - \overline{\overline{y}}_{..})^2$ – C.F.
Block sum of squares = $1/k \sum_i \boldsymbol{B}_i^2$ – C.F.
Treatment sum of squares = $1/r \sum_j \boldsymbol{T}_j^2$ – C.F.
Error Sum of Squares = Total S.Sq. – block sum of squares – treatments sum of squares

The S.S. due to any factorial effect is obtained on multiplying the square of the effect total by the factor (1/4r), where r is the common replication number. Thus,

S.S due to main effect of A = $[A]^2 / 4r$, where $[A] = [ab] - [b] + [a] - [1]$

S.S due to main effect of B = $[B]^2 / 4r$, where $[B] = [ab] + [b] - [a] - [1]$

S.S due to main effect of AB = $[AB]^2 / 4r$, where $[AB] = [ab] - [b] - [a] + [1]$

Degrees of freedom

For blocks, we have (r – 1) d.f.

For main effects A, B and interaction AB each 1 d.f.

For error 3(r – 1) d.f.

ANOVA table for fixed effect model two factor (2^2) experiment in RBD in r replicates

Sources of variation	d.f.	S.S.	M.S.S.	Variance Ratio 'F'
Blocks(replicates)	r – 1	S^2_R	$s^2_R = S^2_R/(r-1)$	$F_R = s^2_R / s^2_E$
Main effect A	1	$S^2_A = [A]^2 / 4r$	$s^2_A = S^2_A$	$F_A = s^2_A / s^2_E$
Main effect B	1	$S^2_{\mathbf{B}} = [B]^2 / 4r$	$s^2_B = S^2_B$	$F_B = s^2_B / s^2_E$
Interaction A x B	1	$S^2_{AB} = [AB]^2 / 4r$	$s^2_{AB} = S^2_{AB}$	$F_{AB} = s^2_{AB} / s^2_E$
Error	3(r-1)	S^2_E = by subtraction	$s^2_E = S^2_E/[3(r-1)]$	
Total	4r – 1	$\sum_i \sum_j (y_{ij} - \overline{\overline{y}}_{..})^2$		

CONCLUSION

For a certain level of α

If $F_R \leq F_{(r-1), 3(r-1)}$ accept H_0, otherwise reject H_0

If F_A, F_B and $F_{AB} \leq F_{1, 3(r-1)}$, the presence of the factorial effect is accepted otherwise rejected.

2^3 FACTORIAL EXPERIMENT

In 2^3 factorial experiment we consider three factors, say, A,B, and C each at two levels, say, $(a_{0,}a_1)$, $(b_{0,}b_1)$ and $(c_{0,}c_1)$ respectively, so that there are $2^3 = 8$ treatment combinations in all.

The eight treatment combinations in a standard order are

'1', a, b, ab, c, ac, bc, abc

where $1=a_0b_0c_0$, $a= a_1b_0c_0$, $b= a_0b_1c_0$, $ab= a_1b_1c_0$, $c= a_0b_0c_1$, $ac= a_1b_0c_1$, $bc = a_0b_1c_1$, $abc= a_1b_1c_1$

In 2^3 factorial experiment we split up the treatment sum of squares with 7d.f. into orthogonal components corresponding to the three main effects A, B and C, three first order (or two factor) interactions AB, AC and BC and one second order interaction (or three factor interaction) ABC, each carrying 1 d.f.

YATES' TABLE FOR 2^3 FACTORIAL EFFECTS

Factorial effect	Treatment mean								Divisor
	(1)	(a)	(b)	(ab)	(c)	(ac)	(bc)	(abc)	
M	+	+	+	+	+	+	+	+	8
A	-	+	-	+	-	+	-	+	4
B	-	-	+	+	-	-	+	+	4
C	-	-	-	-	+	+	+	+	4
AB	+	-	-	+	+	-	-	+	4
AC	+	-	+	-	-	+	-	+	4
BC	+	+	-	-	-	-	+	+	4
ABC	-	+	+	-	+	-	-	+	4

MAIN EFFECTS AND INTERACTIONS

Following the same notations for treatment totals and treatment means as in 2^2 factorial experiment, the simple effect of A,(say), is given by the difference in the mean yields of A as a result of increasing the factor A from the level a_0 to a_1 at other levels of the factors B and C.

Level of B	Level of C	Simple effect of A
b_0	c_0	$(a_1\, b_0\, c_0) - (a_0\, b_0\, c_0) = (a) - (1)$
b_1	c_0	$(a_1\, b_1\, c_0) - (a_0\, b_1\, c_0) = (ab) - (b)$
b_0	c_1	$(a_1\, b_0\, c_1) - (a_0\, b_0\, c_1) = (ac) - (c)$
b_1	c_1	$(a_1\, b_1\, c_1) - (a_0\, b_1\, c_1) = (abc) - (b)$

The main effect of A is defined as the average of these 4 simple effects. Thus

$A = \frac{1}{4}\,[(abc) - (bc) + (ac) - (c) + (ab) - (b) + (a) - (1)]$

$= \frac{1}{4}\,[\{(abc) + (ac) + (ab) + (a)\} - \{(bc) + (c) + (b) + (1)\}]$

$= \frac{1}{4}\,(a - 1)\,(b+1)\,(c+1)$

Similarly, the main effect of the factors B and C can be obtained as

$B = \frac{1}{4}\,[\{(abc) + (ab) + (bc) + (b)\} - \{(a) + (c) + (ac) + (1)\}]$

$= \frac{1}{4}\,(a + 1)\,(b-1)\,(c+1)$

$C = \frac{1}{4}\,[\{(abc) + (bc) + (ac) + (c)\} - \{(a) + (b) + (ab) + (1)\}]$

$= \frac{1}{4}\,(a + 1)\,(b+1)\,(c-1)$

The interaction effects AB,AC and BC is given by

$AB = \frac{1}{4}\,[\{(abc) - (bc) + (ab) - (b)\} - \{(ac) - (c) + (a) - (1)\}]$

$= \frac{1}{4}\,(a - 1)\,(b-1)\,(c+1)$

$AC = \frac{1}{4}\,(a - 1)\,(b + 1)\,(c-1)$

$BC = \frac{1}{4}\,(a + 1)\,(b - 1)\,(c-1)$

The interaction effect of AB with C i.e., the interaction ABC is given by

$ABC = \frac{1}{4}\,[\,(abc) - (bc) - (ac) + (c)\} - \{(ab) + (b) + (a) - (1)]$

$= \frac{1}{4}\,(a - 1)\,(b-1)\,(c-1)$

Model of 2^2 design

If $y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + (\rho)_i + (\epsilon)_{ijkl}$

$(i,j,k = 0,1 : l = 1,2,\ldots r)$

where μ is the general mean

α_i is the effects of i^{th} level of A

β_j is the effects of j^{th} level of B

γ_k is the effects of k^{th} level of C

$(\alpha\beta)_{ij}$ is the interaction effect of i^{th} level of A with j^{th} level of **B**

$(\alpha\gamma)_{ik}$ is the interaction effect of i^{th} level of A with k^{th} level of **C**

$(\beta\gamma)_{jk}$ is the interaction effect of j^{th} level of B with k^{th} level of **C**

$(\alpha\beta\gamma)_{ijk}$ is the interaction effect of i^{th} level of A with j^{th} level of **B and** k^{th} level of **C**

$(\rho)_i$ is the effect due to i^{th} replicate

$(\epsilon)_{ijkl}$ is the error effect due to chance and are i.i.d. $N(0, \sigma^2)$

The above parameter are subject to the following restrictions

$$\sum_{i=0}^{1} \alpha_i = \sum_{j=0}^{1} \beta_j = \sum_{k=0}^{1} \gamma_k = \sum_{i=0}^{1} \rho_i = 0 \text{ and}$$

the sums

$\Sigma\,(\alpha\beta)_{ij}$, $\Sigma\,(\alpha\gamma)_{ik}$, $\Sigma\,(\beta\gamma)_{jk}$, $\Sigma\,(\alpha\beta\gamma)_{ijk}$ are equal to zero respectively.

Statistical analysis of 2^3 factorial experiment

Null Hypothesis**:**

H_0: The blocks as well as treatments are homogeneous

Alternative Hypothesis**:**

H_1: The blocks as well as treatments are not homogeneous

By Using the Yates' table of divisors and signs of 2^3 factorial experiment the various factorial effect totals can be expressed as mutually orthogonal contrasts of the 8 treatment totals. Thus,

$$[A] = [abc] - [bc] + [ac] - [c] + [ab] - [b] + [a] - [1]$$

$$[AB] = [abc] – [bc] +[ab] – [b] – [ac] + [c] – [a] + [1]$$

And so on

S.S. due to main effect A=$[A]^2$ / 8r with 1 d.f.

S.S. due to first order interaction BC = $[BC]^2$/8r with 1 d.f.

S.S. due to second order interaction ABC = $[ABC]^2$/8r with 1 d.f.

And so on

ANOVA TABLE For 2^3 factorial experiment in r randomized blocks

Sources of variation	d.f.	S.S.	M.S.S.	Variance Ratio ‘F’
Blocks(replicates)	r – 1	S^2_R	$s^2_R = S^2_R/(r-1)$	$F_R = s^2_R/ s^2_E$
Main effect A	1	$S^2_A=[A]^2 / 8r$	$s^2_A= S^2_A$	$F_A = s^2_A/ s^2_E$
Main effect B	1	$S^2_{\mathbf{B}}=[B]^2 / 8r$	$s^2_B= S^2_B$	$F_B = s^2_B/ s^2_E$
Main effect B	1	$S^2_{\mathbf{C}}=[C]^2 / 8r$	$s^2_C= S^2_C$	$F_B = s^2_C/ s^2_E$
First order Interaction A x B	1	$S^2_{AB}=[AB]^2 / 8r$	$s^2_{AB}= S^2_{AB}$	$F_{AB} = s^2_{AB} / s^2_E$
First order Interaction A x C	1	$S^2_{AC}=[AC]^2 / 8r$	$s^2_{AC}= S^2_{AC}$	$F_{AC} = s^2_{AC} / s^2_E$
First order Interaction B x C	1	$S^2_{BC}=[BC]^2 / 8r$	$s^2_{Bc}= S^2_{BC}$	$F_{BC} = s^2_{AB} / s^2_E$
Error	7(r-1)	S^2_E= by subtraction	$s^2_E= S^2_E/[7(r-1)]$	
Total	$(r.2^2 – 1)$ = 8r – 1	$\sum_i \sum_j (y_{ij} - \bar{\bar{y}}_{..})^2$		

Conclusion

For a certain level of α if $F_R \leq F_{(r – 1),7(r-1),}$ accept H_0, otherwise reject H_0 for factorial effects

If F_A, F_B, F_C, F_{AB}, F_{AC}, F_{BC}, $F_{ABC} \leq F_{1,7(r-1)}$ the presence of the factorial effects is accepted otherwise rejected.

CONFOUNDING IN FACTORIAL DESIGN

In factorial experiments as the number of factors and the levels at which they are employed increase, the total number of treatment combinations increases rather rapidly and consequently the block size has to be enlarged. For example, for a 210 factorial experiment a complete factorial will require 1024 units. As a consequence of increase in the block size or handling such a huge experiment the purpose of local control is defeated due to the following two reasons.

1. It is sometimes impracticable to get one complete replicate which are relatively homogeneous and
2. The greater heterogeneity is introduced in the experimental error and reduces the discriminating power of the test of significance.

In order to maintain homogeneity within blocks the experimenter must either cut down the number of factors or use an incomplete design. In order to avoid that one the concept of confounding is introduced.

Confounding definition: confounding may be defined as the technique of reducing the size of replication over a no. of blocks at the cost of loosing some information on some effects which is not of much practical importance.

Confounding in 2^3 experiment

In 2^3 experiment, the eight treatment combinations required 8 units of homogeneous material each to form a block. If we decide to use blocks of 4 units each then a full replication will require only two blocks.

In this case 8 treatment combinations are divided into groups of 4 treatments each in a special way so as to confound any one of the less important interactions with blocks and these groups are allocated at random inn the two blocks**.**

For example, let us consider confounding the highest order interaction ABC. The interaction effect ABC is given by

$$ABC = \frac{1}{4} [(abc) - (bc) - (ac) + (c) - (ab) + (b) + (a) - (1)]$$

$$= \frac{1}{4} [(abc) + (a) + (b) + (c) - (ab) + (bc) - (ac) - (1)]$$

In order to confound the interaction ABC with blocks all the treatment combinations with positive sign are allocated at random in one block and those with negative signs in another block. This arrangement gives ABC confounded with blocks and hence we loose less information on ABC.

ABC CONFOUNDED WITH BLOCKS

Replicates	**Block I**	**(1)**	**(ab)**	**(ac)**	**(bc)**
	Block II	**(abc)**	**(a)**	**(b)**	**(c)**

For carrying out the statistical analysis the various factorial effects and their sum of squares are estimated in the usual manner but the sum of squares due to confounded interaction is not computed and it is also not included in the ANOVA table.

Sources of variation	d.f.
blocks	2r - 1
A	1
B	1
C	1
AB	1
AC	1
BC	1
Error	6(r – 1)
total	8r – 1

The error sum of squares is obtained as usual

1.e., S^2_E = T.S.S. - S^2_A - S^2_B - S^2_C - S^2_{AB}- S^2_{AC} - S^2_{BC} - S^2_{ABC}

s