UNIT IV

LATIN SQUARE DESIGN

In RBD whole of the experimental area is divided into relatively homogeneous groups (blocks) and treatments are allocated at random to units within each block, i.e., randomisation was subjected to one restriction i.e., within blocks. A useful method of eliminating fertility variations consists in an experimental layout which will control variation in two perpendicular directions. Such a layout is a Latin Square Design (LSD).

LAYOUT OF THE DESIGN

In this design the number of treatments is equal to the number of replications. Thus in the case of m treatments there have to be m x m = m^2 experimental units. The whole of the experimental area is divided into m^2 experimental units (plots) arranged in a square so that each row as well as each column contains m units (plots). The m treatments are then allocated at random to these rows and columns in such a way that every treatment then occurs once and only once in each row and in each column. Such a layout is known as m x m Latin Square Design (LSD).

STANDARD LATIN SQUARE

A Latin square in which the treatments say A, B, C…occur in the first row and first column in alphabetical order is called a Standard Latin Square Design or Latin Square in Canonical form.

For 2 x 2 and 3 x 3 Latin square design only one standard square exists.

A	B
B	A

A	B	C
B	C	A
C	A	B

For 4 x 4 Latin Square Design 4 standard squares are possible as given in figures.

A	B	C	D
B	C	D	A
C	D	A	B
D	A	B	C

A	B	C	D
B	D	A	C
C	A	D	B
D	C	B	A

A	B	C	D
B	A	D	C
C	D	B	A
D	C	A	B

A	B	C	D
B	A	D	C
C	D	A	B
D	C	B	A

4 X 4 Standard Latin Square Designs

From a standard Latin square design we can generate a number of Latin Squares by permutating the rows, columns and treatments which are known as transformation sets of Latin Squares. The number of squares that can be generated from a standard m x m Latin squares by permutating the rows, columns and letters(treatments) is $(m!)^3$. These are not necessarily all different. If all the rows except the first and all columns are permuted, we generate [(m!) (m – 1)!] squares.

Total number of positive Latin squares of order m x m = (m!) x (m – 1)! x (number of standard squares)

m	Number of standard squares(k)	(m!) (m – 1)!	Total number of Latin squares (m!) (m – 1)! x k
3	1	6x2= 12	12
4	4	24x6=144	576
5	56	120x24=2880	161280
6	9408	720x120=86400	812851200

Statistical Analysis of m x m LSD for one observation per experimental unit;

Let y_{ijk} (i, j, k = 1,2…m) denote the response from the ith row, jth column and receiving the kth treatment.

The linear additive model is

$$y_{ijk} = \mu_{ijk} + \epsilon_{ijk}$$

$$= \mu + (\mu_{i..} - \mu) + (\mu_{.j.} - \mu) + (\mu_{..k} - \mu) + (\mu_{ijk} - \mu_{i..} - \mu_{.j.} - \mu_{..k} + 2\mu)$$

$$= \mu + \alpha_i + \beta_j + \tau_k + \epsilon_{ijk}$$

Efficiency of LSD relative to RBD

Case 1 Rows of LSD as Blocks

Let $s_{E'}^2$ be the error mean sum of squares for RBD with rows of LSD as blocks.

Then the efficiency of LSD relative to RSD is given by

$$E1 = \frac{s_{E'}^2}{s_E^2}$$

6	9408	$720 \times 120 = 86250$	81,285 1200

Advantages of Latin Square Design (L.S.D)

* with two way grouping or Stratification L.S.D controls more of the variation than C.R.D or R.B.D

* L.S.D is an incomplete 3-way layout. Its advantage over the complete 3-way layout is that insted of m^3 experimental units of m^2 units are needed.

* The statistical analysis ~~of~~ is simple though slightly complicated than for R.B.D Even with 1 or 2 missing observations the analysis remains relatively simple.

* More than one factor can be investigated simultaneously and with fever trails than more complicated design.

Disadvantages of L.S.D

* The fundamental assumption than there is no interaction between the three factors of variation (i.e the factors act independently) may not be true in general.

* Unlike R.B.D, in L.S.D the number of treatments is restricted to the number of replications and this limits its field of application.

* In case of missing plots, when several units are missing the statistical analysis becomes quite complex.

* In the field layout, R.B.D is much easy to manage than L.S.D since the former can be performed equally well on a square or retangular field.

Statistical Analysis of $m \times m$ L.S.D for one observation per Experimental Unit

Let y_{ijk} $(i, j, k = 1, 2, \ldots, m)$ denote the response from the unit (plot, in field experimentation) in the i^{th} row, j^{th} column and receiving the k^{th} treatment.

The linear additive model is

$$y_{ijk} = \mu_{ijk} + \epsilon_{ijk}$$

$$= \mu + (\mu_{i..} - \mu) + (\mu_{.j.} - \mu) + (\mu_{..k} - \mu) + (\mu_{ijk} - \mu_{i..} - \mu_{.j.} - \mu_{..k} + 2\mu)$$

$$= \mu + \alpha_i + \beta_j + \tau_k + \epsilon_{ijk}$$

where μ = constant mean effect

$\alpha_i = \mu_{i..} - \mu$ = constant effects due to the i^{th} row

$\beta_j = \mu_{.j.} - \mu$ = " " " " j^{th} column

$\tau_k = \mu_{..k} - \mu$ = " " " k^{th} treatment

ϵ_{ijk} = error term due to unspecified causes.

Least Square Estimates of parameters 5M

The least square estimates of the $(3m+1)$ parameters μ, α_i, β_j and τ_k $(i, j, k = 1, 2, \ldots m)$ in are obtained by minimising the residual sum of squares E given by

$$E = \sum_{i,j,k}^{m} \epsilon_{ijk}^2 = \sum_{i,j,k}^{m} (y_{ijk} - \mu - \alpha_i - \beta_j - \tau_k)^2$$

According to the principle of least squares, the normal equations for estimating μ, α_i, β_j and τ_k are given by

$$\frac{\partial E}{\partial \mu} = 0 = \sum_{(i,j,k)\in S} (y_{ijk} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j - \hat{\tau}_k)$$

$$\frac{\partial E}{\partial \alpha_i} = 0 = \sum_{(j,k)\in S} (y_{ijk} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j - \hat{\tau}_k)$$

$$\frac{\partial E}{\partial \beta_j} = 0 = \sum_{(i,k)\in S} (y_{ijk} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j - \hat{\tau}_k)$$

$$\frac{\partial E}{\partial \tau_k} = 0 = \sum_{(i,j)\in S} (y_{ijk} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j - \hat{\tau}_k)$$

The mean values of α_i, β_j and τ_k are adjusted to be 0. (i.e)

$$\sum_{i=1}^{m} \alpha_i = \sum_{i=1}^{m} (\mu_{i..} - \mu) = 0$$

$$\sum_{j=1}^{m} \beta_j = \sum_{j=1}^{m} (\mu_{.j.} - \mu) = 0$$

and $$\sum_{k=1}^{m} \tau_k = \sum_{k=1}^{m} (\mu_{..k} - \mu) = 0$$

Estimate of μ

$$\frac{\partial E}{\partial \mu} = -2 \sum_{i,j,k} (y_{ijk} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j - \hat{\tau}_k) = 0$$

$$\Rightarrow \sum_{i,j,k} y_{ijk} - \sum_{i,j} \hat{\mu} - \sum_i \hat{\alpha}_i - \sum_j \hat{\beta}_j - \sum_k \hat{\tau}_k = 0$$

$$\sum y_{...} - m^2 \hat{\mu} = 0 \quad \because \sum \hat{\alpha}_i = \sum \hat{\beta}_j = \sum \hat{\tau}_k = 0$$

$$\Rightarrow \hat{\mu} = \frac{1}{m^2} \sum y_{ijk}$$

$$\Rightarrow \hat{\mu} = \bar{y}_{...} \quad \text{——— (1)}$$

Estimate of $\hat{\alpha}_i$

$$\frac{\partial E}{\partial \alpha_i} = -2 \sum_{i,j,k} (y_{ijk} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j - \hat{\tau}_k) = 0$$

$$\sum y_{ijk} - \sum_i \hat{\mu} - \sum_i \hat{\alpha}_i = 0$$

$$\sum y_{ijk} - m\hat{\mu} - m\hat{\alpha}_i = 0$$

$$m\hat{\alpha}_i = \sum y_{ijk} - m\hat{\mu}$$

$$\hat{\alpha}_i = \bar{y}_{i..} - \bar{y}_{...} \quad \text{——— (2)}$$

Estimate of $\hat{\beta}_j$

$$\frac{\partial E}{\partial \beta_j} = -2 \sum_{i,j,k} (y_{ijk} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j - \hat{\tau}_k) = 0$$

$$\sum y_{ijk} - \sum_j \hat{\mu} - \sum_j \hat{\beta}_j = 0$$

$$\sum y_{ijk} - m\hat{\mu} - m\hat{\beta}_j = 0$$

$$m\hat{\beta}_j = \sum y_{ijk} - m\hat{\mu}$$

$$\hat{\beta}_j = \bar{y}_{.j.} - \bar{y}_{...} \quad \text{——— (3)}$$

Estimate of τ_k

$$\frac{\partial E}{\partial \tau_k} = -2 \sum_{i,j,k} (y_{ijk} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j - \hat{\tau}_k) = 0$$

$$\sum y_{ijk} - \sum_k \hat{\mu} - \sum_k \hat{\tau}_k = 0$$

$$\sum y_{ijk} - m\hat{\mu} - m\hat{\tau}_k = 0$$

$$m\hat{\tau}_k = \sum y_{ijk} - m\hat{\mu}$$

$$\hat{\tau}_k = \bar{y}_{..k} - \bar{y}_{...} \quad \text{--- (4)}$$

Null Hypotheses

$H_{0\alpha}$: $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$,

$H_{0\beta}$: $\beta_1 = \beta_2 = \cdots = \beta_m = 0$ and

$H_{0\tau}$: $\tau_1 = \tau_2 = \cdots = \tau_m = 0$

Alternative Hypotheses

$H_{1\alpha}$: At least two α_i's are different

$H_{1\beta}$: At least two β_j's are different

$H_{1\tau}$: At least two τ_k's are different.

Sum of Squares of Calculation

$G = y_{...}$ = Total of all the m^2 observations

$R_i = y_{i..}$ = Total of the m observations in

$C_j = y_{.j.}$ = Total of the m observations in the j^{th} column

$T_k = y_{..k}$ = Total of the m observations from k^{th} treatment

then heuristically, we have

$$\sum_{i,j,k \in S} (y_{ijk} - \bar{y}_{...})^2 = \sum_{i,j,k \in S} \Big[(\bar{y}_{i..} - \bar{y}_{...}) + (\bar{y}_{.j.} - \bar{y})$$

$$+ (\bar{y}_{..k} - \bar{y}_{...}) + (y_{ijk} - \bar{y}_{i..} - \bar{y}_{.j.} - \bar{y}_{..k} + 2\bar{y}$$

$$= m\sum_i (\bar{y}_{i..} - \bar{y}_{...})^2 + m\sum_j (\bar{y}_{.j.} - \bar{y}_{...})^2 + m\sum_k (\bar{y}_{..k} - \bar{y}_{...})^2$$

$$+ \sum_{i,j,k \in S} (y_{ijk} - \bar{y}_{i..} - \bar{y}_{.j.} - \bar{y}_{..k} + 2\bar{y}_{...})^2$$

The product terms vanish, since the algebraic sum of deviations from mean is zero

$\therefore$ T.S.S = S.S.R + S.S.C + S.S.T + S.S.E

where T.S.S is the total sum of squares and S.S.R., S.S.C., S.S.T and S.S.E represent sum of squares due to rows, columns, treatments and error respectively, given by

$$T.S.S = \sum_{i,j,k \in S} (y_{ijk} - \bar{y}_{...})^2; \quad S.S.R = S_R^2 = m\sum_i (\bar{y}_{i..} - \bar{y}_{..}$$

$$S.S.C = S_C^2 = m\sum_j (\bar{y}_{.j.} - \bar{y}_{...})^2; \quad S.S.T = S_T^2 = m\sum_k (\bar{y}_{..k} - \bar{y}_{..}$$

$S.S.E = S_E^2 = T.S.S - S.S.R - S.S.C - S.S.T$

ANOVA Table for $m \times m$ L.S.D

Source of variation	d.f	S.S	M.S.S	Variance Ratio 'F'
Rows	$m-1$	S_R^2	$s_R^2 = S_R^2/(m-1)$	$F_R = s_R^2 / s_E^2$
columns	$m-1$	S_C^2	$s_C^2 = S_C^2/(m-1)$	$F_C = s_C^2 / s_E^2$
Treatment	$m-1$	S_T^2	$s_T^2 = S_T^2/(m-1)$	$F_T = s_T^2 / s_E^2$
Error	$(m-1)(m-2)$	S_E^2	$s_E^2 = S_E^2/(m-1)(m-2)$	
Total	m^2-1			

Conclusion

Let $F_\alpha = F_\alpha\{(m-1), (m-1)(m-2)\}$ be the tabulated value of F for $[(m-1), (m-1)(m-2)]$ d.f at the level of significance 'α'

Thus if $F_R \leq F_\alpha$, we accept $H_{0\alpha}$, otherwise reject $H_{0\alpha}$

if $F_C \leq F_\alpha$, we accept $H_{0\beta}$, otherwise reject $H_{0\beta}$

if $F_T \leq F_\alpha$, we accept $H_{0\tau}$, otherwise reject $H_{0\tau}$

Estimation of one missing value fo L.S.D

Let us suppose that in $m \times m$ Latin square, the observation occurring in the i^{th} row, j^{th} coloumn and receiving the k^{th} treatment is missing. Let us assume that it's value is x, i.e., $\boxed{y_{ijk} = x}$

	1	2 . . . j . . . m	Total
1	y_{111}	y_{121} . . . y_{1j1} . . . y_{1m1}	
2	y_{212}	y_{222} . . . y_{2j2} . . . y_{2m2}	
⋮	⋮		
i	y_{i1k}	y_{i2k} . . . $\boxed{y_{ijk} = x}$ . . . y_{imk}	$R + x$
⋮	⋮		
m	y_{m1m}	y_{m2m} . . . y_{mjm} . . . y_{mmm}	
Total		$C + x$	$S + x$

R = Total of the known observation in the i^{th} row, (i.e), the row containing 'x'

C = Total of known observations in the j^{th} column (i.e) the column containing 'x'

T = Total of known observations receiving k^{th} treatment, (i.e) total of all known treatment values containing 'x'

S = Total of known observations

$$\left.\begin{array}{l}\text{correction}\\ \text{factor}\end{array}\right\} = CF = \frac{(S+x)^2}{m^2}$$

$$\text{Total S.Sq} = \sum y_{ijk}^2 - CF$$

$$= x^2 + \text{constant w.r. to } x - \frac{(S+x)^2}{m^2}$$

$$S.S.R = \frac{(R+x)^2}{m} + \text{constant w.r.to. } x - \frac{(S+x)^2}{m^2}$$

$$S.S.C = \frac{(C+x)^2}{m} + \text{constant w.r.t. } x - \frac{(S+x)^2}{m^2}$$

$$S.S.T = \frac{(T+x)^2}{m} + \text{constant w.r.t. } x - \frac{(S+x)^2}{m^2}$$

$$E = T.S.S - S.S.R - S.S.C - S.S.T$$

$$= x^2 - \cancel{\frac{(S+x)^2}{m^2}} - \left[\frac{(R+x)^2}{m} - \cancel{\frac{(S+x)^2}{m^2}}\right] - \left[\frac{(C+x)^2}{m} - \frac{(S+x)^2}{m^2}\right]$$

$$- \left[\frac{(T+x)^2}{m} - \frac{(S+x)^2}{m^2}\right] + \text{constant}$$

$$E = x^2 - \frac{(R+x)^2}{m} - \frac{(C+x)^2}{m} - \frac{(T+x)^2}{m} + 2\frac{(S+x)^2}{m^2}$$

we will choose 'x' so as to minimise

$$\Rightarrow 2x - \frac{2(R+x)}{m} - \frac{2(C+x)}{m} - \frac{2(T+x)}{m} + 4\frac{(S+x)}{m^2} = 0$$

$$\Rightarrow 2x - \frac{2}{m}\left[R+x+C+x+T+x\right] + 4\frac{(S+x)}{m^2} = 0$$

$$\Rightarrow 2x - \frac{2}{m}\left[R+C+T+3x\right] + 4\frac{(S+x)}{m^2} = 0$$

$$\frac{2m^2x - 2m\left[R+C+T+3x\right] + 4(S+x)}{m^2} = 0$$

$$2m^2x - 6mx + 4x - 2m(R+C+T) + 4S = 0$$

$\div 2$ $$m^2x - 3mx + 2x - m(R+C+T) + 2S = 0$$

$$m^2x - 3mx + 2x = m(R+C+T) - 2S$$

$$(m^2 - 3m + 2)x = m(R+C+T) - 2S$$

+2, -1 -2

$$(m-1)(m-2)x = m(R+C+T) - 2S$$

$$\hat{x} = \frac{m(R+C+T) - 2S}{(m-1)(m-2)}$$

~~ш~~