UNIT. IV

Methods of estimation.

The important methods of obtaining estimators are

1. Method of Maximum Likelihood estimation
2. Method of Minimum chi-square Variance
3. Method of moments
4. Method of least square
5. Method of Min Chi-square
6. Method of Inverse-probability

1. Method of Maximum Likelihood estimation

The general method of estimation known is the MLE was formulated by Gauss

Likelihood function: Let $x_1, x_2 \ldots x_n$ be a random sample of size n from a population with density function $f(x, \theta)$. Then the likelihood function of sample values $x_1, x_2 \ldots x_n$ usually denoted by

$L = L(\theta)$ is their joint density function and given by

$$L = f(x_1, \theta) f(x_2, \theta) \ldots f(x_n, \theta)$$

$$= \prod_{i=1}^{n} f(x_i, \theta)$$

L gives the relative likelihood that the random variable assume a particular set of values $x_1, x_2 \dots x_n$.

For the given sample $x_1, x_2 \dots x_n$ L becomes a function of the variable θ the parameter.

The principle of MLE consists in finding an estimator for the unknown parameter $\theta = (\theta_1, \theta_2 \dots \theta_k)$ say which maximises the likelihood function $L(\theta)$ for variation in parameter

ie we wish to find

$$\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2 \dots \hat{\theta}_k) \text{ so that}$$

$$L(\hat{\theta}) > L(\theta) \quad \forall \theta \in \Theta \text{ ie}$$

$$L(\hat{\theta}) = \sup L(\theta) \quad \forall \theta \in \Theta$$

Thus if there exists a function

$\hat{\theta} = \hat{\theta}(x_1, x_2 \dots x_n)$ of the sample values which maximise L for variation in θ then $\hat{\theta}$ is to be taken as an estimator of θ. $\hat{\theta}$ is usually called Maximum likelihood estimator MLE. Thus $\hat{\theta}$ is the solution if

$$\boxed{\frac{\partial L}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial^2 L}{\partial \theta^2} < 0} \quad \text{— (1)}$$

Since $L > 0$ and Log L is a non-decreasing function of L log L attain their extreme values (maxima or minima) at the same value of $\hat{\theta}$.

The first of the two equations in ① can be rewritten as

$$\frac{1}{L}\frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{\partial \log L}{\partial \theta} = 0 \quad \text{—— ②}$$

If θ is vector valued parameter then

$\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \ldots \hat{\theta}_k)$ is given by the solution of simultaneous equations.

$$\frac{\partial}{\partial \theta_i} \log L = \frac{\partial}{\partial \theta_i} \log L(\theta_1, \theta_2 \ldots \theta_k) = 0 \quad \text{—— ③}$$

$$i = 1, 2 \ldots k$$

The above 2 equations ② and ③ are referred as likelihood equations for estimating the parameters

Properties of MLE.

The following are the assumptions made known as Regularity conditions

i) The first and second order derivatives namely $\frac{\partial \log L}{\partial \theta}$ and $\frac{\partial^2 \log L}{\partial \theta^2}$ exist and are continuous functions of θ in a range R (including the true value θ_0 of the parameter) for almost all x.

For every θ in R $\left|\frac{\partial}{\partial\theta}\log L\right| < F_1(x)$ and $\left|\frac{\partial^2}{\partial\theta^2}\log L\right| < F_2(x)$ where $F_1(x)$ and $F_2(x)$ are integerable functions over $(-\infty,\infty)$.

ii) The third order derivative $\frac{\partial^3}{\partial\theta^3}\log L$ exist such that $\left|\frac{\partial^3}{\partial\theta^3}\log L\right| < M(x)$ where $E[M(x)] < K$, a positive quantity

iii) for every θ in R

$$E\left(\frac{-\partial^2}{\partial\theta^2}\log L\right) = \int_{-\infty}^{\infty}\left(-\frac{\partial^2}{\partial\theta^2}\log L\right)L\,dx = I(\theta)$$

is finite and non-zero.

iv) The range of integeration is independent of θ. But if the range of integeration depends on θ then $f(x,\theta)$ vanishes

This assumption is to make the differentiation under the integral on θ.

The variance of MLE is given by

$$V(\theta) = \frac{1}{I(\theta)} = \frac{1}{-E\left(\frac{\partial^2\log L}{\partial\theta^2}\right)}$$

1. In a random sampling from Normal population $N(\mu, \sigma^2)$ find the MLE for i) μ when σ^2 is known.
ii) σ^2 when μ

Solution

$X \sim N(\mu, \sigma^2)$ then

$$L = \prod_{i=1}^{n} \left[\frac{1}{\sqrt{2\pi}\sigma} e^{\left(-\frac{1}{2\sigma^2}(x_i-\mu)^2\right]} \right.$$

$$= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{\left(-\sum_{i=1}^{n}(x_i-\mu)^2/2\sigma^2\right)}$$

$$\Rightarrow (2\pi\sigma^2)^{-n/2} e^{-\sum_{i=1}^{n}(x_i-\mu)^2/2\sigma^2}$$

$$\log L = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

Case i)

When σ^2 is known, the likelihood equation for estimating μ is

$$\frac{\partial \log L}{\partial \mu} = 0$$

$$\Rightarrow -\frac{1}{2\sigma^2}\sum_{i=1}^{n} 2(x_i-\mu)(-1) = 0$$

$$\sum_{i=1}^{n}(x_i-\mu) = 0$$

$$\sum_{i=1}^{n} x_i - n\mu = 0$$

$$\hat{\mu} = \frac{\sum x_i}{n} = \bar{x}$$

hence MLE for μ is the sample mean

case ii) when μ is known the likelihood estimating of ~~μ and~~ ~~the~~ σ^2 is

$$\frac{\partial}{\partial \sigma^2} \log L = 0$$

$$-\frac{n}{2} \times \frac{1}{\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (x_i - \mu)^2 = 0$$

$$\Rightarrow -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (x_i - \mu)^2 = 0$$

$$\cancel{+}\frac{n}{\cancel{2}\cancel{\sigma^2}} = \cancel{+} \frac{\sum_{i=1}^{n} (x_i - \mu)^2}{\cancel{2}\sigma^{\cancel{4}\,2}}$$

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2$$

case iii) The likelihood equations for simultaneous ~~equa~~ estimation of μ and σ^2 are

$$\frac{\partial}{\partial \mu} \log L = 0 \qquad \frac{\partial}{\partial \sigma^2} \log L = 0$$

thus giving $\hat{\mu} = \bar{x}$

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \hat{\mu})^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$= s^2$ (sample variance.

there $E(\hat{\mu}) = E(\bar{x}) = \mu$

but $E(\hat{\sigma}^2) = E(s^2)$

$\neq \sigma^2$

$\therefore$ MLE's need not necessarily be unbiased

2. Prove that MLE of the parameter α of a population having density function $\frac{2}{\alpha^2}(\alpha - x)$ $0 < x < \alpha$ for a sample of unit size is $2x$ and x being the sample value. Show that the estimate is biased.

Solution

For a random sample of unit size $(n-1)$ the likelihood function is

$$L(\alpha) = f(x, \alpha) \quad ;$$
$$= \frac{2}{\alpha^2}(\alpha - x) \quad ; \quad 0 < x < \alpha$$

Likelihood fn give

$$\frac{d}{d\alpha}\log L = \frac{d}{d\alpha}\left[\log 2 - 2\log\alpha + \log(\alpha - x)\right] = 0$$

$$\Rightarrow \quad -\frac{2}{\alpha} + \frac{1}{\alpha - x}(1) = 0$$

$$-2(\alpha - x) + \alpha = 0$$

$$-2\alpha + 2x + \alpha = 0$$

$$-\alpha = -2x.$$

$$\alpha = 2x.$$

$\therefore$ MLE of $\hat{\alpha} = 2x$.

$$E(\hat{\alpha}) = E(2x) = 2\int_0^\alpha x\, f(x)\, dx$$

$$= 2\int x \frac{2}{\alpha^2}(\alpha - x)$$

$$= \frac{4}{\alpha^2}\int_0^{\alpha} x(\alpha - x)\,dx$$

$$= \frac{4}{\alpha^2}\left[\frac{\alpha x^2}{2} - \frac{x^3}{3}\right]_0^{\alpha}$$

$$= \frac{4}{\alpha^2}\left[\frac{\alpha^3}{2} - \frac{\alpha^3}{3}\right]$$

$$= \frac{\cancel{4}^{\,2}}{\cancel{\alpha^2}}\,\frac{\alpha^{\cancel{3}}}{\cancel{6}_{\,3}}\Big]$$

$$= \frac{2}{3}\alpha.$$

Since $E(\hat{\alpha}) \neq \alpha$, $\hat{\alpha} = 2\bar{x}$ is not a UBE of α.

3. Find MLE for the parameter λ of a
i) Poisson distribution on the basis of sample of size n. Also find its variance.
ii) Show that sample mean $\bar{x}$ is sufficient for estimating the parameter λ.

Solution:

The probability function of the Poisson distribution with parameter λ is given by

$$P(X = x) = f(x, \lambda)$$

$$= \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad x = 0, 1, 2 \ldots$$

Likelihood functions of random sample $x_1, x_2 \ldots x_n$ of n observation from the population is

$$L = \prod_{i=1}^{n} f(x_i, \lambda) = \frac{e^{-n\lambda}\,\lambda^{\Sigma x_i}}{x_1!\,x_2!\ldots x_n!}$$

$$\log L = -n\lambda + n\bar{x}\log\lambda - \sum_{i=1}^{n}\log(x_i!)$$

The likelihood equation for estimating λ.

$$\frac{\partial}{\partial\lambda}\log L = 0 \qquad -n + \frac{n\bar{x}}{\lambda} = 0$$

$$+\cancel{n} = \frac{+\cancel{n}\bar{x}}{\lambda}$$

$$\Rightarrow \lambda = \bar{x}$$

Thus the MLE for λ is sample mean $\bar{x}$

The variance of estimate is given by

$$\frac{1}{V(\hat{\lambda})} = E\left(-\frac{\partial^2}{\partial\lambda^2}\log L\right)$$

$$= E\left(-\frac{\partial}{\partial\lambda}\left(-n + \frac{n\bar{x}}{\lambda}\right)\right)$$

$$= E\left(-\left(-\frac{n\bar{x}}{\lambda^2}\right)\right)$$

$$= \frac{n}{\lambda^2}E(\bar{x})$$

$$= \frac{n}{\lambda^2} X$$

$$\frac{1}{V(\hat{\lambda})} = \frac{n}{\lambda}$$

$$\therefore V(\hat{\lambda}) = \lambda/n$$

As from the theorem if the sufficient estimator exists, it is a function of the MLE

4 Let $x_1, x_2, \dots x_n$ denote random sample of size n from a uniform population with p.d.f

$$f(x, \theta) = 1; \quad \theta - \frac{1}{2} \le x \le \theta + 1/2 \quad -\infty < \theta < \infty$$

obtain MLE for θ.

Solution:

$$L = L(\theta; x_1, x_2 \dots x_n)$$

$$= \begin{cases} 1, & \theta - \frac{1}{2} \le x_i \le \theta + \frac{1}{2} \\ 0 & \text{elsewhere.} \end{cases}$$

If $x_{(1)}, x_{(2)}, \dots x_{(n)}$ is the ordered sample then.

$$\theta - \frac{1}{2} \le x_{(1)} \le x_{(2)} \le \dots \le x_{(n)} \le \theta + \frac{1}{2}$$

Thus L attains its maximum if

$$\theta - \frac{1}{2} \le x_{(1)} \quad \text{and} \quad x_{(n)} \le \theta + \frac{1}{2}$$

$$\theta \le x_{(1)} + \frac{1}{2} \quad \text{and} \quad x_{(n)} - \frac{1}{2} \le \theta.$$

Hence every statistic $t = t(x_1, x_2 \dots x_n)$ such that

$$x_n - \frac{1}{2} \le t(x_1, x_2 \dots x_n) \le x_{(1)} + 1/2$$

provides MLE for θ.

5 Find the MLE of the parameters α and λ (λ being large) of the distribution

$$f(x; \alpha, \lambda) = \frac{1}{\sqrt{\lambda}} \left(\frac{\lambda}{\alpha}\right)^{\lambda} e^{-\lambda x/\alpha} x^{\lambda - 1}$$

$$0 \le x < \infty, \quad \lambda > 0$$

for large values of λ

$$\psi(\lambda) = \frac{\lambda}{\partial \lambda} \log \sqrt{\lambda} = \log \lambda - \frac{1}{2\lambda}$$

and $\psi'(\lambda) = \frac{1}{\lambda} + \frac{1}{2\lambda^2}$

Solution

Let $x_1, x_2 \dots x_n$ be a random sample of size n from the given population

$$L = \prod_{i=1}^{n} f(x_i, \alpha, \lambda) = \left(\frac{1}{\sqrt{\lambda}}\right)^n \left(\frac{\lambda}{\alpha}\right)^{n\alpha} e^{\left(-\frac{\lambda}{\alpha}\sum_{i=1}^{n} x_i\right)} \prod_{i=1}^{n} (x_i^{\lambda-1})$$

$$\log L = -n \log \sqrt{\lambda} + n\lambda(\log \lambda - \log \alpha) - \frac{\lambda}{\alpha}\sum_{i=1}^{n} x_i + (\lambda-1)\sum_{i=1}^{n} \log x_i$$

G is the geometric mean of $x_1, x_2 \dots x_n$ then

$$\log G = \frac{1}{n}\sum_{i=1}^{n} \log x_i$$

$$n \log G = \sum_{i=1}^{n} \log x_i$$

$$\log L = -n\log\sqrt{\lambda} + n\lambda(\log\lambda - \log\alpha) - \frac{\lambda}{\alpha} n\bar{x} + (\lambda-1) n \log G$$

where G is independent of λ and α

The likelihood equation for the simultaneous estimation of α and λ are

$$\frac{\partial}{\partial \alpha} \log L = 0 \quad \text{—— (1)}$$

$$\frac{\partial}{\partial \lambda} \log L = 0 \quad \text{—— (2)}$$

$$\log L = -n\log\Gamma\lambda + n\lambda(\log\lambda - \log\alpha) - \frac{\lambda}{\alpha}n\bar{x} + (\lambda-1)n\log G$$

$$\frac{\partial}{\partial\alpha}\log L = \frac{-n\lambda}{\alpha} + \frac{\lambda}{\alpha^2}n\bar{x} = 0$$

$$\frac{-n\lambda}{\alpha} + \frac{n\bar{x}}{\alpha^2} = 0$$

$$\frac{+n\lambda}{\alpha} = \frac{+n\lambda\bar{x}}{\alpha^2}$$

$$\frac{\alpha^2}{\alpha} = \bar{x}$$

$$\Rightarrow \hat{\alpha} = \bar{x} \quad \text{--- } (3)$$

$$(2) \Rightarrow \frac{\partial}{\partial\lambda}\log L = -n\left(\log\lambda - \frac{1}{2\lambda}\right) + n\left(1(\log\lambda - \log\alpha) + \lambda \times \frac{1}{\lambda}\right] - \frac{n\bar{x}}{\alpha} + n\log G = 0$$

$$\frac{\partial}{\partial\lambda}\log L = -n\left(\log\lambda - \frac{1}{2\lambda}\right) + n\left[1(\log\lambda - \log\alpha) + \lambda \times \frac{1}{\lambda}\right] - \frac{n\bar{x}}{\alpha} + n\log G = 0$$

$$= -n\log\lambda + \frac{n}{2\lambda} + n\log\lambda - (n\log\alpha) + n - \frac{n\bar{x}}{\alpha} + n\log G$$

$$\frac{n}{2\lambda} + \left(n - n\log\alpha + n\log G - \frac{n\bar{x}}{\alpha}\right) = 0$$

$\div$ by $n \Rightarrow$

$$\frac{1}{2\lambda} + 1 - \log\alpha + \log G - \frac{\bar{x}}{\alpha} = 0$$

from ③ $\hat{\alpha} = \bar{x}$

$$\frac{1}{2\lambda} + \left(1 - \log \bar{x} + \log G - \frac{\bar{x}}{\bar{x}}\right) = 0$$

$$= \frac{1}{2\lambda} + (\log G - \log \bar{x}) = 0$$

$$\Rightarrow 1 + 2\lambda(\log G - \log \bar{x}) = 0$$

$$1 - 2\lambda \log\left(\frac{\bar{x}}{G}\right) = 0$$

$$\Rightarrow \hat{\lambda} = \frac{1}{2\log(\bar{x}/G)}$$

∴ MLE for α and λ are

$$\hat{\alpha} = \bar{x} \quad \text{and} \quad \hat{\lambda} = \frac{1}{2\log(\bar{x}/G)}$$

6. Show that Given sample mean is $\bar{x}$ of a random sample of size n of the distribution

$$f(x,\theta) = \frac{1}{\theta} e^{-x/\theta} \qquad 0 \le x \le 1$$

$$\theta > 0$$

find MLE of θ and variance σ^2/n

Solution

Given that

$$f(x,\theta) = \frac{1}{\theta} e^{-x/\theta} \qquad 0 \le x \le 1$$

the likelihood fn is

$$L = \prod f(x_i, \theta)$$

$$= (1/\theta)^n e^{-\Sigma x_i/\theta}$$

$$\log L = -n \log \theta - \Sigma x_i/\theta$$

$$\frac{\partial \log L}{\partial \theta} = \frac{-n}{\theta} + \frac{\Sigma x_i}{\theta^2} = 0$$

$$-n\theta + n\bar{x} = 0$$

$$\bar{x} = \theta$$

$$\therefore E(\bar{x}) = \theta$$

hence it is UBE of θ

$$\frac{\partial \log L}{\partial \theta} = -n/\theta + \frac{\Sigma x_i}{\theta^2} \neq 0$$

$$-n\theta + \Sigma x_i \theta^{-2}$$

$$\frac{\partial \log L}{\partial \theta} = -\frac{n}{\theta} + \frac{\Sigma x_i}{\theta^2}$$

$$= -\frac{n}{\theta^2}(-1) - \frac{2\Sigma x_i}{\theta^3}$$

$$= \frac{n}{\theta^2} - \frac{2\Sigma x_i}{\theta^3}$$

$$= \frac{n}{\theta^2} - \frac{2n\bar{x}}{\theta^3}$$

$$= \frac{-2n\bar{x} + n\theta}{\theta^3}$$

$$= \frac{-2n\bar{x} + n\bar{x}}{(\bar{x})^3}$$

$$= \frac{-n\bar{x}}{(\bar{x})^3} = \frac{-n}{(\bar{x})^2} < 0$$

$\therefore$ MLE of θ with variance is given by

$$V(\hat{\theta}) = \frac{1}{E\left(-\frac{\partial^2 \log L}{\partial \theta^2}\right)}$$

$$= \frac{1}{\left(-\left(\frac{-n}{(\bar{x})^2}\right)\right)} \qquad \bar{x} = \theta^2$$

$$= \frac{1}{n/\theta^2}$$

$$= \theta^2/n$$

7. Find the MLE of θ for the given density function

$$f(x,\theta) = \theta e^{-\theta x} \qquad \theta \geq 0$$

Solution

Given $f(x_8, \theta) = \theta e^{-\theta x}$

The likelihood fn is

$$L = \prod f(x_i, \theta)$$

$$= (\theta)^n e^{-\theta \Sigma x_i}$$

$$\log L = n \log \theta - \theta \Sigma x_i$$

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - \Sigma x_i = 0$$

$$\Sigma x_i = \frac{n}{\theta}$$

$$n\bar{x} = n/\theta$$

$$\bar{x} = 1/\theta$$

$$\frac{\partial^2 \log L}{\partial \theta^2} = -\frac{n}{\theta^2}$$

$$= \frac{-n}{(1/\bar{x})^2}$$

$$= -n\bar{x}^2 < 0$$

the variance is given by

$$V(\bar{x}) = \frac{1}{E\left(-\frac{\partial^2 \log L}{\partial \theta^2}\right)}$$

$$= \frac{1}{E(-(-n\bar{x}^2))}$$

$$= \frac{1}{n\bar{x}^2} = \frac{1}{n(1/\theta)^2} = \frac{\theta^2}{n}$$

$$\therefore V(\bar{x}) = \theta^2/n$$

Method of moments

This is the simplest method of obtaining the estimate of the unknown parameters of the population. It is the oldest method introduced by Karl Pearson and hence known as Pearson's method of moment.

It is based on moments of population also the sample drawn from population.

Suppose we have the population with p.d.f $f(x_i, \theta_1, \theta_2 \ldots \theta_n)$ with k unknown parameters namely $\theta_1, \theta_2 \ldots \theta_k$.

We suppose that $x_1, x_2 \ldots x_n$ is the set of n random observation drawn from the above population to estimate the unknown parameter.

The moment method consists of equating the population raw moment with the sample raw moment and then solving them w.r.t. to unknown parameter where the population raw moments are function of unknown parameter.

Let μ_r' be the r^{th} order raw moment about $x=0$ in the population then the first k moments about origin is given by.

$$\mu_r'(\theta_1, \theta_2 \ldots \theta_k) = \int_{-\infty}^{\infty} x^r f(x, \theta, \theta_2 \ldots \theta_k)\, dx.$$

This k moment are expressed in terms of parameter $\theta_1, \theta_2 \ldots \theta_k$. To find the estimator of this parameter we solve this above k equation for $\theta_1, \theta_2 \ldots \theta_k$ which are expressed in term of $\mu_1', \mu_2' \ldots \mu_k'$ are replaced by $m_1' m_2' \ldots m_k'$.

The first k raw moment are selected from the above population

Properties

1. If $x_1, x_2 \ldots x_n$ are iid observations then by the weak law of large nos

$$\frac{1}{n} \Sigma x_i^r \xrightarrow{P} E(x^r)$$

$$\Rightarrow m_r' \xrightarrow{P} \mu_r'$$

$$\Rightarrow \bar{x} \rightarrow \mu.$$

The moment method estimate are consistent

2. Moment method estimate are unbiased

3. They are not in general efficient

4. They are distributed the asymptotically normal.

5. The moment method estimate are identical with that of MLE when the form of the distn is exponential

i $f(x) = \exp[b + b_1 x + b_2 x^2 + \ldots]$

where b_i are independent of x but may depend on θ.

1. Find the estimator of p using method of moments

The binomial distribution is

$$f(x) = \binom{n}{x} p^x q^{n-x} \qquad x = 0, 1, 2 \dots n$$

To estimate p consider 1st raw moment about origin

By def. $\mu_1' = \Sigma x p_x$

$$= \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x}$$

$$= np \,\Sigma \frac{\cancel{x}(n-1)!}{\cancel{x}(x-1)!\,(n-x)!} p^{x-1} q^{n-x}$$

$$= np\,(q+p)^{n-1}$$

$$\mu_1' = np$$

$\therefore$ The sample raw moments

$\mu_1' = np$

$\mu_1' = m_1'$

$$p = \frac{m_1'}{n}$$

$$\hat{p} = \frac{m_1'}{n}$$

2. Obtain the estimate of the parameter θ in the poison population using method of moments

Solution:

Consider the Poison population with the form

$$p(x) = \frac{e^{-\theta} \theta^{x}}{x!}$$

$$\mu_1' = \Sigma x \, p x$$

$$= \sum_{x=0}^{n} \frac{x e^{-\theta} \theta^{x}}{x (x-1)!}$$

$$= e^{-\theta} \left[\theta + \frac{\theta^2}{1!} + \frac{\theta^3}{2!} + \cdots \right]$$

$$= e^{-\theta} \theta \left[1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \cdots \right]$$

$$= \cancel{e}^{-\theta} \theta \cancel{e}^{\theta}$$

$$= \theta$$

Now we equate the 1^{st} raw moments of the sample

$$\mu_1' = \bar{x}$$

we have

$$\mu_1' = m_1$$

$$\hat{\theta} = \bar{x}$$

3. Estimate the parameter in normal distribution using method of moments

Solution :

The prob density function of normal distribution is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty \le x \le \infty$$

$$\sigma^2 > 0$$

$$\mu_1' = \int_{-\infty}^{\infty} x\, f(x, \mu, \sigma^2)\, dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$=$$

$$\vdots$$

$$\mu_1' = \mu$$

$$m_1' = \mu$$

$$\therefore \hat{\mu} = m_1'$$

$$\sigma^2 = \mu_2' - (\mu_1')^2$$

$$\mu_2' = \int x^2 f(x, \mu, \sigma^2)\, dx$$

$$= \int (x - \mu + \mu)^2 f(x, \mu, \sigma^2)\, dx$$

$$= \int_{-\infty}^{\infty} [(x-\mu)^2 + 2(x-\mu)\mu + \mu^2]\, f(x, \mu, \sigma^2)\, dx$$

$$= \int [(x-\mu)^2 + \mu^2 + 2x\mu - 2\mu^2]\, f(x, \mu, \sigma^2)\, dx$$

$$= \int ([x-\mu)^2 + \mu^2)\, f(x, \mu, \sigma^2)\, dx + 2\mu \int x\, f(x, \mu, \sigma^2) dx$$
$$- 2\mu^2 \int f(x, \mu, \sigma^2)$$

$$= \int [(x-\mu)^2 + \mu^2\, f(x, \mu, \sigma^2)\, dx + 2\mu\mu - 2\mu^2$$

$$= \int \left((x-\mu)^2 + \mu^2 \right) f(x, \mu, \sigma^2)\, dx + 2\mu^2 - 2\mu^2$$

$$= \int (x-\mu)^2 f(x)\, dx + \int \mu^2 f(x)\, dx$$

$$\mu_2' = \sigma^2 + \mu^2$$

$$\mu_2' = \sigma^2 + (\mu_1')^2$$

$$\Rightarrow \sigma^2 = \mu_2' - (\mu_1')^2$$

3. Estimate α and β in the case of Pearson's type 3 distribution by the method of moments

$$f(x, \alpha, \beta) = \frac{\beta^\alpha}{\Gamma\alpha} x^{\alpha-1} e^{-\beta x} \quad 0 < x < \infty$$

Solution

consider the p.d.f. we have

$$\mu_r' = \int x^r f(x)\, dx$$

$$= \int_0^\infty x^r \frac{\beta^\alpha}{\Gamma\alpha} x^{\alpha-1} e^{-\beta x}\, dx$$

$$= \frac{\beta^\alpha}{\Gamma\alpha} \int_0^\infty x^{r+\alpha-1} e^{-\beta x}\, dx$$

multiply and divide by $\Gamma\alpha+r \cdot \beta^{\alpha+r}$

$$= \frac{\beta^\alpha}{\Gamma\alpha} \frac{\Gamma\alpha+r}{\beta^{\alpha+r}} \frac{\beta^{\alpha+r}}{\Gamma\alpha+r} \int_0^\infty x^{r+\alpha-1} e^{-\beta x}\, dx$$

$$= \frac{\Gamma \alpha+r}{\Gamma \alpha \, \beta^{r}} \int_0^{\infty} \underbrace{\frac{\beta^{\alpha+r}}{\Gamma \alpha+r} \, x^{\alpha+r-1} \cdot e^{-\beta x} \, dx}_{\text{value} = 1}$$

$$\mu_r' = \frac{\Gamma \alpha+r}{\Gamma \alpha \, \beta^{r}}$$

If $r = 1$

$$\mu_1' = \frac{\Gamma \alpha+1}{\beta \Gamma \alpha}$$

$$= \frac{\alpha !}{\beta (\alpha-1)!} = \frac{\alpha (\cancel{\alpha-1})!}{\beta (\cancel{\alpha-1})!}$$

$$= \frac{\alpha}{\beta} \quad \text{--- } 1$$

If $r = 2$

$$\mu_2' = \frac{\Gamma \alpha+2}{\Gamma \alpha \, \beta^{2}}$$

$$= \frac{(\alpha+1)!}{(\alpha-1)! \, \beta^{2}}$$

$$= \frac{(\alpha+1)(\alpha+1-1)(\alpha+1-2)!}{(\alpha-1)! \, \beta^{2}}$$

$$= \frac{\alpha(\alpha+1)}{\beta^{2}}$$

$$\frac{\mu_2'}{\mu_1'^2} = \frac{\alpha(\alpha+1)}{\beta^2} \Big/ \left(\frac{\alpha}{\beta}\right)^2$$

$$= \frac{\alpha(\alpha+1)}{\beta^2} \cdot \frac{\beta^2}{\alpha^2}$$

$$\frac{\mu_2'}{(\mu_1')^2} = \frac{\alpha+1}{\alpha}$$

$$\mu_2'\alpha = \mu_1'^2(\alpha+1)$$

$$\mu_2'\alpha = \alpha\mu_1'^2 + \mu_1'^2$$

$$\alpha\mu_2' - \alpha\mu_1'^2 = \mu_1'^2$$

$$\alpha(\mu_2' - \mu_1'^2) = \mu_1'^2.$$

$$\hat{\alpha} = \frac{\mu_1'^2}{\mu_2' - \mu_1'^2}$$

using equation (1)

$$\mu_1' = \frac{\alpha}{\beta}$$

$$\beta = \frac{\alpha}{\mu_1'},$$

$$\beta = \frac{\mu_1'^2 / \mu_2' - \mu_1'^2}{\mu_1'}$$

$$= \frac{\mu_1'}{\mu_2' - \mu_1'^2}$$

hence

$$\hat{\lambda} = \frac{m_1'^2}{m_2' - m_1'^2}$$

$$\hat{p} = \frac{m_1'}{(m_2' - m_1'^2)}$$

where m_1' and m_2' are the sample moment.

4. For the double poisson distribution

$$p(x) = P(X=x) = \frac{1}{2}\frac{e^{-m_1} m_1^x}{x!} + \frac{1}{2}\frac{e^{-m_2} m_2^x}{x!}$$

$x = 0, 1, 2 \ldots$

show that the estimates for m_1 and m_2 by the method of moments are

$\mu_1' \pm \sqrt{\mu_2' - \mu_1' - \mu_1'^2}$

Solution:

$$\mu_1' = \sum_{x=0}^{\infty} x\, p(x)$$

$$= \frac{1}{2}\sum_{x=0}^{\infty} x \frac{e^{-m_1} m_1^x}{x!} + \frac{1}{2}\sum_{x=0}^{\infty} x \frac{e^{-m_2} m_2^x}{x!}$$

$$= \frac{1}{2}m_1 + \frac{1}{2}m_2 \quad \text{— (1)} \qquad 2\mu_1' = m_1 + m_2$$

since the first and second summations are the means of Poisson distributions with parameters m_1 and m_2 resp

$$\mu_2' = \sum_{x=0}^{\infty} x^2 p(x)$$

$$= \frac{1}{2}\sum_{x=0}^{\infty} x^2 \frac{e^{-m_1} m_1^x}{x!} + \frac{1}{2}\sum_{x=0}^{\infty} x^2 \frac{e^{-m_2} m_2^x}{x!}$$

$$= \frac{1}{2}(m_1^2 + m_1) + \frac{1}{2}(m_2^2 + m_2)$$

$$= \frac{1}{2}(m_1^2 + m_1 + m_2^2 + m_2)$$

$$= \frac{1}{2}(m_1 + m_2 + m_1^2 + m_2^2)$$

$$= \frac{1}{2}[2\mu_1' + m_1^2 + m_2^2) \quad \text{——— } (2)$$

$$= \frac{1}{2}[2\mu_1' + m_1^2 + (2\mu_1' - m_1)^2]$$

$$= \frac{1}{2}[2\mu_1' + m_1^2 + 4\mu_1'^2 + m_1^2 - 4\mu_1' m_1]$$

$$\mu_2' = \frac{1}{2}[2\mu_1' + 2m_1^2 + 4\mu_1'^2 - 4\mu_1' m_1]$$

$$0 = \mu_1' + m_1^2 + 2\mu_1'^2 - 2\mu_1' m_1 - \mu_2'$$

$$\Rightarrow \underbrace{m_1^2}_{a} - \underbrace{2\mu_1' m_1}_{b} + \underbrace{2\mu_1'^2 + \mu_1' - \mu_2'}_{c} = 0$$

The above eqn is in quadratic form

$$m_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2\mu_1') \pm \sqrt{4\mu_1'^2 - 4(2\mu_1'^2 + \mu_1' - \mu_2')}}{2(1)}$$

$$= \frac{2\mu_1' \pm \sqrt{4\mu_1'^2 - 4(2\mu_1'^2 + \mu_1' - \mu_2')}}{2}$$

$$= \frac{2\mu_1' \pm 2\sqrt{\mu_1'^2 - 2\mu_1'^2 - \mu_1' + \mu_2'}}{2}$$

$$= \mu_1' + \sqrt{\mu_1'^2 - 2\mu_1'^2 - \mu_1' + \mu_2'}$$

$$m_1 = \mu_1' \pm \sqrt{\mu_2' - \mu_1' - \mu_1'^2}$$

Now consider equation (2).

$$\mu_2' = \frac{1}{2}\left(m_1 + m_2 + m_1^2 + m_2^2\right)$$

as $m_1 = 2\mu_1' - m_2$ $\Rightarrow m_1 + m_2 = 2\mu_1'$

$$= \frac{1}{2}\left[2\mu_1' + (2\mu_1' - m_2)^2 + m_2^2\right]$$

$$= \frac{1}{2}\left[2\mu_1' + 4\mu_1'^2 + 2m_2^2 - 4\mu_1' m_2\right]$$

$$\mu_2' = \left[\mu_1' + 2\mu_1'^2 + m_2^2 - 2\mu_1' m_2\right]$$

$$0 = \underbrace{m_2^2}_{a} - \underbrace{2\mu_1' m_2}_{b} + \underbrace{2\mu_1'^2 + \mu_1' - \mu_2'}_{c}$$

$$m_2 = \frac{-(-2\mu_1') \pm \sqrt{(-2\mu_1')^2 - 4(2\mu_1'^2 + \mu_1' - \mu_2')}}{2(1)}$$

$$= \frac{2\mu_1' \pm \sqrt{4\mu_1'^2 - 4(2\mu_1'^2 + \mu_1' - \mu_2')}}{2}$$

$$= \frac{\cancel{2}\left[\mu_1' \pm \sqrt{(\mu_1'^2 - 2\mu_1'^2 - \mu_1' + \mu_2')}\right]}{\cancel{2}}$$

$$m_2 = \mu_1' \pm \sqrt{\mu_2' - \mu_1' - \mu_1'^2}$$

$$\therefore \hat{m}_1 = \mu_1' \pm \sqrt{\mu_2' - \mu_1' - \mu_1'^2}$$

and

$$\hat{m}_2 = \mu_1' \pm \sqrt{\mu_2' - \mu_1' - \mu_1'^2}$$

Method of Minimum Chi-square method

when the given sample is large in size we can estimate the unknown parameter of the population using the sample by the Maximum likelihood method but in some situation it may be found difficult to apply in such situation we can prefer min χ^2 method

The method of min χ^2 make use of Pearson's chi-square statistic.

Let $f_1, f_2 \ldots f_k$ be the observed frequencies in k groups or classes and unknown probabilities that f_i elements belong to i^{th} group or class be $P_i (i = 1, 2 \ldots k)$ p_i's are the function of unknown parameter $\theta_1, \theta_2 \ldots \theta_m$.

Thus $P_i = P_i(\theta)$ where $\theta = \theta_1, \theta_2 \ldots \theta_m$

Suppose the total sample size is n

$\therefore \Sigma f_i = n$

The expected frequencies are $np_1(\theta)$, $np_2(\theta) \cdots np_k(\theta)$

We know Pearsonian χ^2 statistic is

$$\chi^2 = \sum_{i=1}^{k} \frac{[f_i - np_i(\theta)]}{np_i(\theta)}$$

Under the method of minimum χ^2 one has to choose $(\theta_1, \theta_2 \cdots \theta_n)$ which minimises. This will be minimum when $np_i(\theta)$ is as close as possible to f_i.

So to obtain the estimates of θ_i's partially differential χ^2 w.r.t to θ_i $(i=1,2 \cdots n)$ successively and equate them to zero. Also check that second order derivat are non-ve.

ie $$\frac{\partial \chi^2}{\partial \theta_i} = 0 \quad \text{for } i = 1,2 \cdots m \text{ and}$$

$$\frac{\partial^2 \chi^2}{\partial \theta_i^2} \geq 0$$

If $\frac{\partial \chi^2}{\partial \theta_i} = 0$ provides simultaneous equat in m unknown one get the estimated values of $\theta_1 \theta_2 \cdots \theta_m$ respectively