UNIT-II

Definition :-

Efficiency :-

The estimates which are confined to unbiasedness will in general lead to ~~estim~~ existence of more than one consistent estimator of a parameter. Hence there is a necessity of a criterion to choose between the estimators with common property of consistency. Such a criterian which is based on variance of the sampling distribution of estimators is known as efficiency.

If T_1 and T_2 are two consistent estimator of certain parameter θ we have.

$V(T_1) < V(T_2)$ for all n then T_1 is more efficient than T_2 for all sample size.

Example: Consider normal distribution in sampling from normal population $N(\mu, \sigma^2)$ where σ^2 is known and sample mean $\bar{x}$ is unbiased and consistent estimator.

By symmetry sample median (Md) is also unbiased and also it is consistent as $V(Md) = \frac{1}{4n} 2\pi\sigma^2$

$= \pi\sigma^2/2n$

$V(Md) \longrightarrow 0$ as $n \to \infty$ hence it is consistent estimator of μ.

hence to find out best among two estimators ($\bar{x}$ and Md) efficiency is used.

$$\forall n \quad V(\bar{x}) = \frac{\sigma^2}{n}$$

$$\text{for large } V(Md) = \frac{\pi\sigma^2}{2n} = 1.57(\sigma^2/n)$$

as $V(\bar{x}) < V(Md)$ we conclude sample mean is more efficient estimator of μ than sample median

Most Efficient Estimator :

If T_1 is the most efficient estimator with variance V_1 and T_1 is any other estimator with variance V_2, then the efficiency E of T_2 is defined as:

$$E = \frac{V_1}{V_2}$$

Obviously, E cannot exceed unity

Example :-

A random sample $(x_1, x_2, x_3, x_4, x_5)$ of size 5 is drawn from a normal population with unknown mean μ.

Consider the following estimators to estimate μ.

(i) $t_1 = \dfrac{x_1 + x_2 + x_3 + x_4 + x_5}{5}$ (ii) $t_2 = \dfrac{x_1 + x_2}{2} + x_3$

(iii) $t_3 = \dfrac{2x_1 + x_2 + \lambda x_3}{3}$

Where λ is such that t_3 is an unbiased estimator of μ.

Find λ Are t_1 and t_2 unbiased? State giving reasons, the estimator which is best among t_1, t_2 & t_3

Solution:

$E(x_i) = \mu$, $var(x_i) = \sigma^2$, $Cov(x_i, x_j) = 0$

$(i \neq j = 1, 2 \cdots n)$

(i) $E(t_1) = E\left(\frac{x_1 + x_2 + x_3 + x_4 + x_5}{5}\right)$

$= \frac{1}{5} E(x_1 + x_2 + x_3 + x_4 + x_5)$

$= \frac{1}{5} E(\Sigma x_i)$

$= \frac{1}{5} \cancel{5} E(x_i)$

$= \frac{1}{5} \cancel{5} \mu.$

$\boxed{E(t_1) = \mu.}$

(ii) $E(t_2) = E\left(\frac{x_1 + x_2}{2}\right) + E(x_3)$

$= \frac{1}{2} E(x_1 + x_2) + E(x_3)$

$= \frac{1}{2} \cdot (\mu + \mu) + \mu.$

$= \frac{1}{2} \cdot \cancel{2}\mu + \mu.$

$\boxed{E(t_2) = 2\mu.}$

t_2 is not an unbiased estimator of μ.

(ii) $E(t_3) = \mu$. (t_3 is unbaiased estimator of μ)

$\Rightarrow \frac{1}{3} E(2x_1 + x_2 + \lambda x_3) = \mu$.

$\Rightarrow \frac{1}{3} 2E(x_1) + E(x_2) + \lambda E(x_3) = \mu$.

$\Rightarrow 2\mu + \mu + \lambda\mu = 3\mu$.

$\Rightarrow \cancel{3\mu} + \lambda\mu = \cancel{3\mu}$.

$\lambda\mu = 0$

$\boxed{\lambda = 0}$

$V(t_1) = \frac{1}{25}\{V(x_1) + V(x_2) + V(x_3) + V(x_4) + V(x_5)\}$

$= \frac{1}{25}\{\sigma^2 + \sigma^2 + \sigma^2 + \sigma^2 + \sigma^2\}$

$= \frac{1}{\cancel{25}_5} \times \cancel{5}\sigma^2$

$\boxed{V(t_1) = \frac{1}{5}.\sigma^2}$

$V(t_2) = \frac{1}{4}\{V(x_1) + V(x_2)\} + V(x_3)$

$= \frac{1}{\cancel{4}_2} \times \cancel{2}\sigma^2 + \sigma^2$

$\frac{1}{2}\sigma^2 + \sigma^2$

$\frac{\sigma^2}{2} + \sigma^2$

$= \frac{3}{2}\sigma^2$

$V(t_3) = \frac{1}{9}\{4V(x_1) + V(x_2)\}$

$= \frac{1}{9}\{4\sigma^2 + \sigma^2\} \Rightarrow \frac{5\sigma^2}{9}$

$V(t_1)$ is least, t_1 is the best estimator of μ.

Example: 2 X_1, X_2 and X_3 is a random sample size 3 from a population with mean value μ and variance σ^2. T_1, T_2, T_3 are the estimators used to estimate mean value μ, where (i) $T_1 = X_1 + X_2 - X_3$ (ii) $T_2 = 2X_1 + 3X_3 - 4X_2$ and (iii) $T_3 = \frac{1}{3}(\lambda X_1 + X_2 + X_3)$

(i) Are T_1 and T_2 unbiased estimators?

(ii) Find the value of λ such that T_3 is unbiased estimator of μ.

(iii) With this value of λ is T_3 a consistent estimator?

(iv) Which is the best estimator?

Solution:-

Since X_1, X_2, X_3 is a random sample from a population with mean μ and variance σ^2

$$E(X_i) = \mu, \quad Var(X_i) = \sigma^2 \quad \text{and}$$

$$Cov(X_i, X_j) = 0, \quad (i \neq j = 1, 2, \dots n)$$

(i) $E(T_1) = E(X_1 + X_2 - X_3)$

$= E(X_1) + E(X_2) - E(X_3)$

$= \mu + \cancel{\mu} - \cancel{\mu}$

$\boxed{E(T_1) = \mu}$

(ii) $E(T_2) = E(2X_1 - 4X_2 + 3X_3)$

$= 2E(X_1) - 4E(X_2) + 3E(X_3)$

$= 2\mu - 4\mu + 3\mu.$

$\boxed{E(T_2) = \mu}$

$E(T_3) = \mu.$

(iii) $E(T_3) = E\left[\frac{1}{3}(\lambda X_1 + X_2 + X_3)\right]$

$\mu = \frac{1}{3}\lambda E(X_1) + E(X_2) + E(X_3)$

$\mu = \frac{1}{3}\lambda\mu + \mu + \mu$

$3\mu = \lambda\mu + 2\mu.$

$\mu = \lambda\mu.$

$\boxed{1 = \lambda}$

With $\lambda = 1$, $T_3 = \frac{1}{3}(x_1 + x_2 + x_3) = \bar{x}$. Since sample mean is a consistent estimator of population mean μ, by weak Law of Large Numbers, T_3 is a consistent estimator of μ.

(iv)

$Var(T_1) = Var(x_1) + Var(x_2) + Var(x_3) = 3\sigma^2$

$Var(T_2) = 4\,Var(x_1) + 16\,Var(x_2) + 9\,Var(x_3) = 29\sigma^2$

$Var(T_3) = \frac{1}{9}\left[Var(x_1) + Var(x_2) + Var(x_3)\right] = \frac{1}{3}\sigma^2$

$V(T_3)$ is minimum, T_3 is best estimator.

Minimum Variance Unbiased Estimators :

If a statistic $T = T(x_1, x_2, \cdots x_n)$ based on sample of size n is such that :

(i) T is unbiased for $\gamma(\theta)$, for all $\theta \in \Theta$ and

(ii) It has the smallest variance among the class of all unbiased estimators of $\gamma(\theta)$, the T is called the minimum variance unbiased estimator of $\gamma(\theta)$

$$E_\theta(T) = \gamma(\theta) \text{ for all } \theta \in \Theta$$

$$Var_\theta(T) \leq Var_\theta(T') \text{ for all } \theta \in \Theta$$

Theorem : An M-V.U is Unique in the ~~concerning MVU estimators.~~ sence that if T_1 and T_2 are MVU estimators for $\gamma(\theta)$, then $T_1 = T_2$ almost surely.

Solution :

$$E_{(\theta)}(T_1) = E_\theta(T_2) = \gamma(\theta), \text{ for all } \theta \in \Theta.$$

$$Var_\theta(T_1) = Var(T_2), \text{ for all } \theta \in \Theta$$

Consider a new estimator, $T = \frac{1}{2}(T_1 + T_2)$ which is also unbiased since

$$E(T) = \frac{1}{2}\{E(T_1) + E(T_2)\} = \gamma(\theta)$$

$$Var(T) = Var\left\{\frac{1}{2}(T_1 + T_2)\right\}$$

Variance is a square !

$(a+b)^2 = a^2 + b^2 + 2ab$

$$= \frac{1}{4}\{Var\, T_1 + Var(T_2)\}$$

$$= \frac{1}{4}\left\{ var(T_1) + Var(T_2) + 2\,cov(T_1, T_2)\right\}$$

$$= \frac{1}{4}\left\{ var(T_1) + Var(T_2) + 2\rho\sqrt{var(T_1)\,var(T_2)}\right\}$$

$$= \frac{1}{4}\left\{ var(T_1) + var(T_1) + 2\rho\sqrt{var(T_1)}^2 \right\}$$ $\because var(T_1) = var(T_2)$

$$= \frac{1}{2}\,var(T_1)\,(1+\rho)$$

$$var(T) = \frac{1}{2}\,var(T_1)\,(1+\rho)$$

ρ is khaul pearson's co-efficient of correlation between T_1 and T_2.

since T_1 is the MVU estimator, $Var(T) \geq var(T_1)$

substitute $V(T)$ value. in above.

$$\Rightarrow \frac{1}{2}\,\cancel{var(T_1)}\,(1+\rho) \geq \cancel{var(T_1)}$$

$$\Rightarrow \frac{1}{2}(1+\rho) \geq 1$$

$$\boxed{\rho \geq 1}$$

$|\rho| \leq 1$, We must have $\rho = 1$, T_1 and T_2 must have a linear relation

$$T_1 = \alpha + \beta T_2.$$

Where α and β are constants independent of $x_1, x_2, \dots x_n$ but may depend on θ. We may have

$\alpha = \alpha(\theta)$,

$\beta = \beta(\theta)$

Taking expectation on both side in linear equation

$$T_1 = \alpha + \beta T_2$$

$\therefore E(T_1) = E(T_2) = \theta$.

$$E(T_1) = \alpha + \beta E(T_2)$$

$$\boxed{\theta = \alpha + \beta\theta.}$$

$$Var(T_1) = Var(\alpha + \beta T_2)$$

$$= \beta^2 Var(\alpha + T_2)$$

$$= \beta^2 Var(T_2)$$

$$\boxed{1 = \beta^2 \Rightarrow \beta = \pm 1}$$

But since $\rho(T_1, T_2) = +1$, the co-efficient of regression of T_1 on T_2 must be positive

$$\boxed{\beta = 1} \Rightarrow \boxed{\alpha = 0}$$

substitute in linear equation

$$T_1 = \alpha + \beta T_2$$

$$T_1 = 0 + 1(T_2)$$

$$\boxed{T_1 = T_2}$$

1) Such that for a sample size 'n' drawn from a normal population with mean μ and variance σ^2. The statistic $\hat{\mu} = \frac{1}{n+1} \sum_{i=1}^{n} x_i$ is the most efficient for estimating 'μ' though is biased.

Solution:

$$E(\hat{\mu}) = \frac{1}{n+1} E\left[\sum x_i\right]$$

$$= \frac{1}{n+1} \sum E(x_i)$$

$$= \frac{1}{n+1} n\mu.$$

$$= \frac{n}{n+1} \cdot \mu.$$

$$\therefore E(\hat{\mu}) \neq \mu.$$

($\therefore \hat{\mu}$ is a biased estimator of μ)

Now,

$$V(\hat{\mu}) = V\left(\frac{1}{n+1} \sum x_i\right)$$

$$= \left(\frac{1}{n+1}\right)^2 V(\sum x_i)$$

$$\boxed{\frac{\sum x_i}{n} = \bar{x} \qquad n\bar{x} = \sum x_i}$$

$$= \left(\frac{1}{n+1}\right)^2 \cdot V(n\bar{x})$$

$$= \left(\frac{n}{n+1}\right)^2 \cdot V(\bar{x}) \longrightarrow (1)$$

$$= \left(\frac{n}{n+1}\right)^2 \cdot \sigma^2/n$$

Then,

$V(\hat{\mu}) <$

$\Rightarrow V(\hat{\mu}) < V(\bar{x}) \longrightarrow ②$

With Respect to!

$$V(x \text{ median}) = \frac{\pi \sigma^2}{2n}$$

$$V(\bar{x}) / V(x\text{median}) = \frac{\sigma^2/n}{\pi\sigma^2/2n}$$

$$= 2/\pi < 1$$

$$\Rightarrow V(\bar{x}) < V(x \text{ median}) \rightarrow ③$$

From ② & ③ we Conclude that $\hat{\mu}$ is most efficient estimator of μ.

2) Consider a random sample of size 'n' from a normal population with mean μ and variance σ^2. Suggest any two consistant estimator μ and hence identify an efficient estimator.

Solution:

$$X \sim N(\mu, \sigma^2)$$

$$\bar{X} \sim N(\mu, \sigma^2/n)$$

$$(X\text{med}) \sim N(\mu, \pi\sigma^2/2n)$$

$$E(\bar{x}) = \mu \longrightarrow ①$$

$V(\bar{x}) = \sigma^2/n$, $V(\bar{x}) = 0$ as $n \to \alpha$ — ②

From ① & ② we say that $\bar{x}$ is a Consistant estimator of μ.

(x med) $\sim N(\mu, \pi\sigma^2/2n)$ and

$E(x \text{ med}) = \mu \longrightarrow$ ③

$V(x \text{ med}) = \pi\sigma^2/2n$

$V(x \text{ med}) = 0$ as $n \to \alpha \longrightarrow$ ④

From ③ & ④ we say that

(x median) is a Consistant estimator of

$V(\bar{x}) = \sigma^2/n$, $V(x \text{ median}) = \pi\sigma^2/2n$

$$\frac{V(\bar{x})}{V(x \text{ median})} < 1 \Rightarrow V(\bar{x}) < V(x \text{ median})$$

$\bar{x}$ is an efficient estimator of μ.

x median $\Rightarrow \dfrac{V(\bar{x})}{V(x \text{ med})} = \dfrac{\sigma^2/n}{\pi\sigma^2/n}$

$= 2/\pi < 1$

$\Rightarrow 0.6364 < 1$

Let T_1 & T_2 be unbiased estimators of $g(\theta)$ with co-efficient e_1 and e_2 respectively and $\rho = \rho_\theta$ be correlation co-efficient between them, then

$$\sqrt{e_1 e_2} - \sqrt{(1-e_1)(1-e_2)} \le \rho \le \sqrt{e_1 e_2} + \sqrt{(1-e_1)(1-e_2)}$$

Solution:

Let T be minimum variance unbiased estimator of $g(\theta)$

As given T_1 & T_2 are unbiased estimator.

$$E(T_1) = g(\theta) = E(T_2) \quad \forall \ \theta \in \Theta$$

$$e_1 = \frac{V_\theta(T)}{V_\theta(T_1)} = \frac{V}{V_1}$$

$$\Rightarrow V_1 = \frac{V}{e_1}$$

$$e_2 = \frac{V_\theta(T)}{V_\theta(T_2)} = \frac{V}{V_2}$$

$$\Rightarrow V_2 = \frac{V}{e_2}$$

Now consider another estimator

$$T_3 = \lambda T_1 + \mu T_2$$

which is also unbiased estimator

$$E(T_3) = E(\lambda T_1 + \mu T_2)$$

$$= (\lambda + \mu) \cdot E(T_1 + T_2)$$

$E(T_3) = (\lambda + \mu) \cdot g(\theta)$

$V(T_3) = V(\lambda T_1 + \mu T_2)$

$= \lambda^2 \cdot V(T_1) + \mu^2 \cdot V(T_2) + 2\lambda\mu \, \mathrm{Cov}(T_1, T_2)$

$= V\left(\frac{\lambda^2}{e_1} + \frac{\mu^2}{e_2} + 2\lambda\mu \frac{\rho}{\sqrt{e_1 e_2}}\right)$

$V(T_3) \geq V$ as V is min variance.

$\therefore \frac{\lambda^2}{e_1} + \frac{\mu^2}{e_2} + \frac{2\rho\lambda\mu}{\sqrt{e_1 e_2}} \geq 1$

$\Rightarrow \frac{\lambda^2}{e_1} + \frac{\mu^2}{e_2} + \frac{2\rho\lambda\mu}{\sqrt{e_1 e_2}} \geq (\lambda + \mu)^2$

$\frac{\lambda^2}{e_1} + \frac{\mu^2}{e_2} + \frac{2\rho\lambda\mu}{\sqrt{e_1 e_2}} \geq \lambda^2 + \mu^2 + 2\lambda\mu$

$\frac{\lambda^2}{e_1} + \frac{\mu^2}{e_2} + \frac{2\rho\lambda\mu}{\sqrt{e_1 e_2}} - \lambda^2 - \mu^2 - 2\lambda\mu \geq 0$

$\lambda^2\left(\frac{1}{e_1} - 1\right) + \mu^2\left(\frac{1}{e_2} - 1\right) + 2\lambda\mu\left(\frac{\rho}{\sqrt{e_1 e_2}} - 1\right) \geq 0$

$\left(\frac{1}{e_1} - 1\right)\lambda^2 + \left(\frac{1}{e_2} - 1\right)\mu^2 + 2\lambda\mu\left(\frac{\rho}{\sqrt{e_1 e_2}} - 1\right) \geq 0$

$\left(\frac{1}{e_1} - 1\right) \cdot \frac{\lambda^2}{\mu^2}$

$\left(\frac{1}{e_1} - 1\right)\left(\frac{\lambda}{\mu}\right)^2 + \frac{2\lambda\mu}{\mu^2}\left(\frac{\rho}{\sqrt{e_1 e_2}} - 1\right) + \left(\frac{1}{e_2} - 1\right)\frac{\mu^2}{\mu^2} \geq 0$

$$\left(\frac{1}{e_1}-1\right)\left(\frac{\lambda}{\mu}\right)^2+\frac{2\lambda}{\mu}\left(\frac{\rho}{\sqrt{e_1 e_2}}-1\right)+\left(\frac{1}{e_2}-1\right)\geq 0 \rightarrow (2)$$

In general $e_i < 1 \Rightarrow \frac{1}{e_i} > 1$ (or) $\left(\frac{1}{e_i}-1\right) > 0$

$\forall\ i = 1, 2 \ldots$

(2) is in quadratic form if

$Ax^2 + Bx + C \geq 0$, $\forall\ A > 0$, $C > 0$ and

if $B^2 - 4AC \leq 0 \rightarrow (3)$

using (3) we get (2) as

$$2^2\left(\frac{\rho}{\sqrt{e_1 e_2}}-1\right)^2 - 4\left(\frac{1}{e_1}-1\right)\left(\frac{1}{e_2}-1\right) \leq 0$$

$$\left(\frac{\rho-\sqrt{e_1 e_2}}{e_1 e_2}\right)^2 - \left(\frac{(1-e_1)}{e_1}\ \frac{(1-e_2)}{e_2}\right) \leq 0$$

$$\left(\rho-\sqrt{e_1 e_2}\right)^2 - (1-e_1)(1-e_2)$$

$$\rho^2 - 2\sqrt{e_1 e_2}\,\rho + e_1 e_2 - (1 - e_2 - e_1 + e_1 e_2)$$

$$\rho^2 - 2\sqrt{e_1 e_2}\,\rho + \cancel{e_1 e_2} - 1 + e_2 + e_1 - \cancel{e_1 e_2}$$

$$\Rightarrow \rho^2 - 2\sqrt{e_1 e_2}\,\rho + (e_1 + e_2 + 1) = 0$$

$Ax^2 + Bx + C$ $\quad \therefore$ here $\boxed{x = \rho}$

Roots of the equation :-

$\boxed{a = 1}$, $\boxed{b = -2\sqrt{e_1 e_2}}$, $\boxed{c = (e_1 + e_2 + 1)}$

$$\Rightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \frac{2\sqrt{e_1 e_2} \pm \sqrt{(-2\sqrt{e_1 e_2})^2 - 4(1)\times(e_1 + e_2 + 1)}}{2(1)}$$

$$\Rightarrow \frac{\cancel{2}\sqrt{e_1 e_2} \pm \cancel{2}\sqrt{e_1 e_2 - (e_1 + e_2 + 1)}}{\cancel{2}}$$

$$\Rightarrow \sqrt{e_1 e_2} \pm \sqrt{e_1 e_2 - (e_1 + e_2 + 1)}$$

$$\Rightarrow \sqrt{e_1 e_2} \pm \sqrt{e_1 e_2 - e_1 - e_2 - 1}$$

$$\Rightarrow \sqrt{e_1 e_2} \pm \sqrt{e_2(e_1 - 1) - 1(e_1 - 1)}$$

$$\Rightarrow \sqrt{e_1 e_2} \pm \sqrt{(e_2 - 1)(e_1 - 1)}$$

$$\therefore \sqrt{e_1 e_2} - \sqrt{(e_1 - 1)(e_2 - 1)} \leq \rho \leq \sqrt{e_1 e_2} + \sqrt{(e_1 - 1)(e_2 - 1)}$$

$$\Rightarrow \sqrt{e_1 e_2} - \sqrt{(1 - e_1)(1 - e_2)} \leq \rho \leq \sqrt{e_1 e_2}\ \sqrt{(1 - e_1)(1 - e_2)}$$

Theorem: If T_1 is a MVUB estimator of $\gamma(\theta)$ $\theta \in \Theta$ and T_2 is any other unbiased estimators of $\gamma(\theta)$ with efficiency $e < 1$ then no unbiased linear combination of T_1 and T_2 can be an MVUE of $\gamma(\theta)$.

Proof:

The linear combination

$$T = l_1 T_1 + l_2 T_2$$

will be unbiased estimator of $\gamma(\theta)$, if

$$E(T) = l_1 E(T_1) + l_2 E(T_2) = \gamma(\theta) \Rightarrow l_1 + l_2 = 1$$

since $E(T_1) = \gamma(\theta) = E(T_2)$

We have $$e = \frac{Var(T_1)}{Var(T_2)}$$

$$Var(T_2) = \frac{Var(T_1)}{e}$$

$$\rho = \rho(T_1, T_2) = \sqrt{e}$$

$$Var(T) = l_1^2 var(T_1) + l_2^2 Var(T_2) + 2 l_1 l_2 Cov(T_1, T_2)$$

$$= l_1^2 var(T_1) + l_2^2 var(T_2) + 2 l_1 l_2 \rho \sqrt{Var(T_1) Var(T_2)}$$

$$= l_1^2 var(T_1) + l_2^2 \frac{Var(T_1)}{e} + 2 l_1 l_2 \sqrt{Var(T_1) \cdot \frac{Var(T_1)}{e}}$$

$$= Var\, T_1 \left(l_1^2 + \frac{l_2^2}{e} + 2 l_1 l_2 \frac{\rho}{\sqrt{e}} \right.$$

$$= Var(T_1) \left(l_1^2 + 2 l_1 l_2 \frac{\sqrt{e}}{\sqrt{e}} + \frac{l_2^2}{e} \right.$$

$= \text{Var}\, T_1 \left(l_1^2 + 2 l_1 l_2 + \frac{l_2^2}{e} \right)$ $\quad \therefore 0 < e < 1$

$\Rightarrow \frac{1}{e} > 1$

$\therefore \text{Var}\, T > \text{Var}\, T_1 \left(l_1^2 + 2 l_1 l_2 + l_2^2 \right)$ $\quad \because (a+b)^2 = a^2 + b^2 + 2ab$

$\text{Var}\, T > \text{Var}\, T_1 \left(l_1 + l_2 \right)^2$

$\text{Var}\, T > \text{Var}(T_1)$

$\therefore T$ cannot be MVUE

<u>Example:</u> If T_1 and T_2 be two unbiased estimator of $\gamma(\theta)$ with Variance σ_1^2 & σ_2^2 and correlation ρ what is the best linear combination of T_1 & T_2. and what is its variance.

<u>Solution:</u>

Let T_1 & T_2 be two unbiased estimator of $\gamma(\theta)$.

$$E(T_1) = E(T_2) = \gamma(\theta) \longrightarrow (1)$$

Let T be a linear combination of T_1 & T_2 given by $\boxed{T = l_1 T_1 + l_2 T_2}$, where l_1 & l_2 are constant.

$$E(T) = l_1 E(T_1) + l_2 E(T_2)$$

$$= (l_1 + l_2)\, \gamma(\theta)$$

$\therefore T$ is also an UBE of $\gamma(\theta)$ iff $\boxed{l_1 + l_2 = 1}$

now,

$$V(T) = V(l_1 T_1 + l_2 T_2)$$

$$= l_1^2 V(T_1) + l_2^2 V(T_2) + 2 l_1 l_2 \operatorname{cov}(T_1, T_2)$$

$$= l_1^2 \sigma_1^2 + l_2^2 \sigma_2^2 + 2 l_1 l_2 \sigma_1 \sigma_2 \longrightarrow ③$$

We want the minimum value

Hence we use minimum principle

$$\frac{\partial}{\partial l_1} V(T) = 0 \Rightarrow 2 l_1 \sigma_1^2 + 2 l_2 \sigma_1 \sigma_2 \rho = 0$$

$$\Rightarrow l_1 \sigma_1^2 + l_2 \rho \sigma_1 \sigma_2 = 0$$

$$\frac{\partial}{\partial l_2} V(T) = 0 \Rightarrow 2 l_2 \sigma_2^2 + 2 l_1 \sigma_1 \sigma_2 \rho = 0$$

$$\Rightarrow l_2 \sigma_2^2 + l_1 \rho \sigma_1 \sigma_2$$

subtraction eq,

$$\Rightarrow l_1 \sigma_1^2 + l_2 \rho \sigma_1 \sigma_2 - l_2 \sigma_2^2 - l_1 \rho \sigma_1 \sigma_2$$

$$\Rightarrow l_1 (\sigma_1^2 - \rho \sigma_1 \sigma_2) - l_2 (\sigma_2^2 - \rho \sigma_1 \sigma_2)$$

$$\Rightarrow l_1 (\sigma_1^2 - \rho \sigma_1 \sigma_2) = l_2 (\sigma_2^2 - \rho \sigma_1 \sigma_2)$$

$$\frac{l_1}{\sigma_2^2 - \rho \sigma_1 \sigma_2} = \frac{l_2}{\sigma_1^2 - \rho \sigma_1 \sigma_2}$$

$$\frac{l_1 + l_2}{\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2} = \frac{1}{\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2}$$

$$\therefore l_1 = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

$$l_2 = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

$\longrightarrow$ (4)

with these

Values of l_1 & l_2 T given by the (4) is the best unbiased linear combination of T_1 & T_2 and its variance is

$$V(T) = l_1^2\sigma_1^2 + l_2^2\sigma_2^2 + 2l_1l_2\rho\sigma_1\sigma_2$$

Sufficiency :-

An estimator is said to be sufficient for a parameter, if it contain all the information in the sample regarding the parameter.

If $T = t(x_1, x_2 \dots x_n)$ is an estimator of a parameter θ, based on sample $x_1, x_2 \dots x_n$ of size n from the population with density $f(x, \theta)$ such that the Conditional distribution of θ, then T is sufficient estimator for θ.

Eg:- Consider the random samples $x_1, x_2 \dots x_n$ belongs to Bernoulli population with parameter 'p', $0 < p < 1$

$$x_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } q = 1-p \end{cases}$$

$$T = t(x_1, x_2 \dots x_n) = x_1 + x_2 + \dots + x_n \sim B(n, p)$$

$$P(T=k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad k = 0, 1, 2 \dots n.$$

The conditional distribution of $(x_1, x_2 \dots x_n)$ given T

is $P(x_1 \cap x_2 \cap \dots x_n / T = k) = \dfrac{P(x_1 \cap x_2 \dots x_n)}{P(T=k)}$

$$= \begin{cases} \dfrac{p^k (1-p)^{n-k}}{\binom{n}{k} p^k (1-p)^{n-k}} = \dfrac{1}{\binom{n}{k}} \\ 0 \qquad \text{if } \sum_{i=1}^{n} x_i \neq k \end{cases}$$

Since this does not depend 'p' $\therefore$ T is sufficient.

Neymans Factorisation theorem :-

Necessary and sufficient condition for a distribution to admit sufficient statistic is provide by the factorisation theorem due to Neymann.

Statement : $T = t(x)$ is sufficient for θ iff the joint density function $L(x_a y)$ of the sample values can be expressed in the form.

$$L = g_\theta\,(t(x)\; h(x))$$

Where, $g_\theta\,[t(x)]$ depends on θ and x only through the value of $t(x)$ and $h(x)$ independents of θ.

Here, The likelihood function

$$L(x,\theta) = g(\hat{\theta},\theta)\; h(x)$$

Where, $g(\hat{\theta},\theta)$ is a function of $\hat{\theta}$ and θ

$h(x)$ is a function independent of the parameter θ.

Invariance property of ~~of~~ sufficient estimators:

If T is a sufficient estimator for the parameter θ and if $\psi(T)$ is one to one function of T and $\psi(T)$ is sufficient for $\psi(\theta)$.

Examples :-

1) Let $x_1, x_2 \ldots x_n$ be a random sample from a population with pdf.

$$f(x,\theta) = \theta x^{\theta-1} \quad ; \quad 0 < x < 1 \;,\; \theta > 0$$

show that $t_1 = \prod_{i=1}^{n}$ is sufficient for θ.

Solution:

$$f(x,\theta) = \theta x^{\theta-1} \quad 0<x<1, \ \theta>0$$

We know that,

$$L(x_i,\theta) = \prod_{i=1}^{n} f(x_i,\theta)$$

$$L(x_i,\theta) = f(x_1,\theta). f(x_2,\theta) \cdots f(x_n,\theta)$$

$$= \theta x_1^{\theta-1} . \theta x_2^{\theta-2} \cdots \theta x_n^{\theta-1}$$

$$= \theta^n \prod_{i=1}^{n} x_i^{\theta-1}$$

$$= \theta^n \prod_{i=1}^{n} x_i^{\theta} . \prod_{i=1}^{n} x_i^{-1}$$

$$= \theta^n \prod_{i=1}^{n} x_i^{\theta} . \frac{1}{\prod_{i=1}^{n} x_i}$$

$$L(x_i,\theta) = g \prod_{i=1}^{n} (x_i,\theta) . h(x_i)$$

$$= g(t_1,\theta) . h(x_i)$$

Hence, by factorisation theorem $t_1 = \prod_{i=1}^{n} x_i$ is sufficient estimator of θ.

2) Let $x_1, x_2 \ldots x_n$ be a random sample from $N(\mu,\sigma^2)$ population. Find sufficient estimator for μ and σ^2

Solution:

Let $\theta = (\mu, \sigma^2)$; $-\infty < \mu < \infty$, $0 < \sigma^2 < \infty$

Solution:

The p.d.f of uniform distribution is

$$f_\theta(x) = \begin{cases} \frac{1}{\theta}, & 0 \le x_i \le \theta \\ 0, & \text{otherwise} \end{cases}$$

Let $k(a,b) = \begin{cases} 1 & \text{if } a \le b \\ 0 & \text{if } a > b \end{cases}$

Then $f_\theta(x_i) = \dfrac{k(0, x_i)\, k(x_i, \theta)}{\theta}$

We know that

$$L(x_i, \theta) = \prod_{i=1}^{n} f_\theta(x_i)$$

$$= \frac{k(0, x_1)\, k(x_1, \theta)}{\theta} \cdot \frac{k(0, x_2)\, k(x_2, \theta)}{\theta} \cdots\cdots \frac{k(0, x_n)\, k(x_n, \theta)}{\theta}$$

$$= \prod_{i=1}^{n} \frac{k(0, x_i)\, k(x_i, \theta)}{\theta}$$

$$= \frac{1}{\theta^n}\left[k\left(0, \min_{1 \le i \le n} x_i\right) \cdot k\left(\max_{1 \le i \le n} x_i, \theta\right)\right]$$

$$= g_\theta[t(x)] \cdot h(x)$$

Where $g_\theta[t(x)] = \dfrac{k(t(x), \theta)}{\theta^n} = \max\limits_{1 \le i \le n} x_i$

$$h(x) = k\left[0, \min_{1 \le i \le n} x_i\right]$$

Hence by factorisation theorem

$T = \text{Max } x_i$ is sufficient estimator for θ

Q. Let $x_1, x_2 \ldots x_n$ be a random sample from a distribution with p.d.f

$$f(x, \theta) = e^{-(x-\theta)}, \quad \theta < x < \infty$$
$$-\infty < \theta < \infty$$

Obtain sufficient statistic for θ.

Solution :-

The p.d.f of the given distribution is,

$$f(x, \theta) = e^{-(x-\theta)}$$

We know that,

$$L(x_i, \theta) = \sum_{i=1}^{n} f(x_i, \theta)$$

$$= e^{-(x_1-\theta)} \cdot e^{-(x_2-\theta)} \cdots e^{-(x_n-\theta)}$$

$$= \sum_{i=1}^{n} e^{-(x_i-\theta)}$$

$$= e^{\sum_{i=1}^{n} x_i} e^{\theta n}$$

$$= e^{-\sum_{i=1}^{n} x_i} \cdot e^{n\theta}$$

Now consider the order statistic

Let $Y_1, Y_2, \ldots Y_n$ denote the order statistic of the random sample such that $Y_1 < Y_2 < \cdots < Y_n$

The p.d.f of the smallest observation is

$$g_1(y_1, \theta) = n\left[1 - F(y_1)\right]^{n-1} f(y_1, \theta)$$

Now,

$$f(x) = \int_{\theta}^{x} e^{-(x-\theta)} \, dx$$

$$= \left. \frac{e^{-(x-\theta)}}{-1} \right|_{\theta}^{x}$$

$$= \frac{e^{-(x-\theta)}}{-1} + e^{-(\theta-\theta)}$$

$$= 1 - e^{-(x-\theta)}$$

$$\therefore g_1(y_1, \theta) = n\left[1 - F(y_1)\right]^{n_1-1} f(y)$$

$$= n\left[e^{-(y_1-\theta)}\right]^{n-1} \cdot e^{-(y_1-\theta)}$$

$$= \begin{cases} n \cdot e^{-n(y_1-\theta)}, & \theta < y_1 < \alpha \\ 0, & \text{otherwise} \end{cases}$$

Thus the likelihood function of $x_1, x_2 \ldots x_n$ may be written as,

$$L = e^{n\theta} \cdot e^{-\sum_{i=1}^{n} x_i}$$

$$= n \cdot e^{-n(y_1 - \omega)} \quad \frac{e^{-\sum_{i=1}^{n} x_i}}{n e^{-ny_1}}$$

$$= g_1(\min x_i, \omega) \quad \frac{e^{-\sum x_i}}{n \cdot e(-n \min x_i)}$$

Hence by crieterian the first order statistic

$y_1 = \min(x_1, x_2 \dots x_n)$ is a sufficient statistic for

θ.