UNIT-I

BASIC CONCEPT

Define - Population :-

⇒ Population is an aggregative of objects, animate or inanimate under study of population. population may be finite or infinite.

⇒ A statistical population is the collection of all possible observations of a specified characteristic of interest.

Eg:- All indian women aged 40-49, All possible pieces of land, all patients suffering from a particular disease.

Sample :-

⇒ Any finite collection containing part of the observations from the population is called a sample. A finite subset of statistical individual in a sample is called the Sample size

Parameter & statistic :-

parameter : Any statistical constant, which is computed by considering each and every observation of the population is calleas parameter. statistical constant of the population are usually refered to as parameter.

Eg:- Mean – μ (population Mean)

Variance – σ^2 (" variance)

Statistic : Any statistical constant which computed by considering a part of the information from the population is called statistic.

statistical measures computed from the sample observations are termed as "statistic".

Eg:- $\bar{x}$, (sample mean)

s, (,, variance)

Statistical Inference:-

⇒ In the statistical procedure used for drawing inferences about the population from the sample data are covered statistical Inference.

Inference can be broadly classified into the following

1) Estimation Theory.

* point estimation
* Interval estimation.

2) Testing of hypothesis.

Estimator:-

* A statistic which is used to estimate an unknown parameter is called as "estimators"

* Eg:- Sample mean $\bar{x}$, is an unbiased estimator of the population mean μ the sample median, may be considered an estimator of the population mean μ.

point Estimation :-

⇒ A point estimate is a "single number", which is used as an estimate of the unknown population parameter.

Eg : Sample mean $\bar{x}$, may be used as an unbiased estimator of the population mean μ.

Interval Estimation :

⇒ In Interval estimation our aim is to find. Confidence limits based on sample observations within. which the unknown parameter lies ~~between~~ within Confidence co-efficient ~~limits based on the Sample obser-vation~~ Computed in probability terms.

⇒ A interval estimation having two numbers or two values which is used as an estimate of the Confidence limits based on the sample observation.

Characteristics of an ideal Estimator :

⇒ Unbiasedness

⇒ Consistency

⇒ Efficiency

⇒ Sufficiency

Unbiasedness :-

A statistic $\hat{Q}$ is called an unbiased estimator of the parameter, "θ" if and only if,

$E(\hat{\theta}) = \theta$. If $E(\hat{\theta}) \neq \theta$ then $\hat{\theta}$ is said to

(theta)

the biased of estimator of the parameter 'θ'. The quantity

$E(\hat{\theta}) - \theta = 0$ is called unbiased of the estimator 'θ'.

If $E(\hat{\theta}) - \theta > 0$, then $\hat{\theta}$ is said to be possitively

biased and if $E(\hat{\theta}) - \theta < 0$, it is negatively biased.

Even if $E(\hat{\theta}) \neq \theta$ it may happen that $\lim_{n \to \alpha} E(\hat{\theta}) = \theta$.

In that case θ is said to be asymptotically unbiased

estimator of the parameter 'θ'

PROBLEMS :-

(1). Let $x_1, x_2, x_3, \ldots\ldots x_n$ are the sample random

Sample from a population then such a sample mean is

a unbiased estimation of population mean. (or) Derive

that sample mean is the unbiased estimator of the

population mean (μ)

(2)

Proof:

Let $\bar{x} = \frac{\Sigma x}{n}$,

$$= \frac{x_1 + x_2 + \cdots + x_n}{n}$$

And μ be the population mean

$\therefore x_1, x_2, x_3 \ldots x_n$ is a random sample from

the population with Mean μ.

$E(x_1) = E(x_2) = \cdots E(x_n) = \mu$ ——— ①

To prove, $E(\bar{x}) = \mu$.

$$E(\bar{x}) = E\left[\frac{x_1 + x_2 + \cdots + x_n}{n}\right]$$

$$= \frac{1}{n} E[x_1 + x_2 + \cdots x_n]$$

=①. $$= \frac{1}{n}[E(x_1) + E(x_2) + \cdots\cdots E(x_n)]$$

$$= \frac{1}{n}(\mu_1 + \mu_2 + \cdots + \mu_n)$$

$$= \frac{1}{n} n\mu$$

$$\boxed{E(\bar{x}) = \mu.}$$

Sample mean is unbiased estimator of the population mean 'μ'.

② If a random variable has a binomial population with parameter 'n' and 'p' such that their sample proportion x/n is an unbiased estimation of P.

Proof.

We know that $E(x) = np$

$$\frac{Ex}{n} = P$$

$$p = E(x/n)$$

Sample ~~population~~ Proportion x/n is an unbiased estimator of parameter p.

(3). If a random sample of size n is selected from normal population with mean μ and var 1. Verify whether $T = 1/n \Sigma x_i^2$ is an unbiased estimator of μ^2+1.

Given $E(x_i) = \mu$, $V(x_i) = 1$

Proof:

$$V(x_i) = E(x_i^2) - [E(x_i)]^2$$

$$1 = E(x_i^2) - \mu^2$$

$$\mu^2 + 1 = \underline{E(x_i^2)} \quad \text{—— ①}$$

$$E(T) = E(1/n \Sigma x_i^2) = 1/n [E(\Sigma x_i^2)]$$

$$= 1/n [\Sigma \underline{E(x_i^2)}]$$

substitute

Eq. equation ①

$$E(T) = 1/n \sum_{i=1}^{n} (\mu^2+1)$$

$$= 1/\cancel{n} \cdot \cancel{n} (\mu^2+1)$$

$$\boxed{E(T) = \mu^2+1}$$

$T = 1/n \Sigma x_i^2$ is an unbiased estimator of μ^2+1.

(4). Let $x_1, x_2, x_3 \dots x_n$ be a random variable having zero-one distribution having value of one with probability p value zero, with prob $1-p$, then $\frac{x(x-1)}{n(n-1)}$ is an unbiased estimator of p^2.

Proof:

Each x_1 has the distribution $B(1,p)$ then x_2 has the distribution $B(n,p)$.

$$E(x)=np, \quad V(x)=npq$$

$$V(x) = E(x^2) - [E(x)]^2$$

$$E(x^2) = V(x) + [E(x)]^2$$

$$E(x^2) = npq + (np)^2$$

$$E(x^2) = npq + n^2p^2 \quad \text{——— (1)}$$

Now,

$$E\left\{\frac{x(x-1)}{n(n-1)}\right\} = \frac{1}{n(n-1)} E[x(x-1)]$$

$$\frac{1}{n(n-1)} E(x^2-x) = \frac{1}{n(n-1)} E[x^2-x]$$

$$= \frac{1}{n(n-1)} Ex^2 - E(x)$$

$$= \frac{1}{n(n-1)} \{npq + n^2p^2 - np\}$$

$$= \frac{1}{n(n-1)} np\{q + np - 1\}$$

$$= \frac{1}{n(n-1)} np\{\cancel{1}-p+np-\cancel{1}\}$$

$$= \frac{1}{n(n-1)} np\{np - p\}$$

$$= \frac{np}{n(n-1)} p(\cancel{n} n-1)$$

$$= \frac{\cancel{n}p^2}{\cancel{n}(\cancel{n-1})} (\cancel{n-1})$$

$$\boxed{\frac{1}{n(n-1)} E(x^2-x) = p^2}$$

$E\left[\frac{x(x-1)}{n(n-1)}\right]$ is an unbiased estimator of p^2

5. A random variable 'x' follows binomial distribution,

$$x = \begin{cases} 0, & \text{with prob } 1-\theta \quad (1) \\ 1, & \text{" " } \theta \quad (2) \end{cases}$$

Verify wheather $\frac{T(T-1)}{n(n-1)}$ is an unbiased estimator of θ^2, where $T = \sum_{i=1}^{n} x_i$ that is T is the sum of all the sample observations.

Proof:

$$E(x) = \Sigma x \cdot p(x) = 0 \times (1-\theta) + (1 \times \theta) = \underset{\text{zero}}{0} + \theta = \theta$$

$$E(x^2) = \Sigma x^2 \cdot p(x) = 0^2(1-\theta) + (1^2 \times \theta) = 0 + \theta = \theta$$

$$V(x) = E(x^2) - [E(x)]^2 = \theta - \theta^2 = \theta(1-\theta)$$

Here, $T = \Sigma x_i$

$E(x_i) = \theta$

$$E(T) = E(\Sigma x_i)$$

$$E(T) = \Sigma E(x_i) = \Sigma \theta = \boxed{n\theta},$$

$$V(T) = V[\Sigma (x_i)]$$

$$= \Sigma V(x_i) = \Sigma \theta(1-\theta) = \boxed{n\theta(1-\theta)}$$

$$V(T) = E(T^2) - [E(T)]^2$$

$$E[T^2] = V(T) + E[(T)]^2$$

$$= n\theta(1-\theta) + (n\theta)^2$$

$$E[T^2] = n\theta(1-\theta) + n^2\theta^2$$

$\frac{T(T-1)}{n(n-1)} \Rightarrow \frac{T^2-T}{n(n-1)} \Rightarrow$ Take 'E' on both side.

$$\frac{E(T^2) - E(T)}{n(n-1)} \quad \text{—— (1)}$$

$$= \frac{n\theta(1-\theta) + n^2\theta^2 - n\theta}{n(n-1)}$$

$$= \frac{\cancel{n\theta} - n\theta^2 + n^2\theta^2 - \cancel{n\theta}}{n(n-1)}$$

$$= \frac{\cancel{n}\theta(-\theta + n\theta)}{\cancel{n}(n-1)}$$

$$= \frac{\theta \cdot \theta(\cancel{n-1})}{(\cancel{n-1})}$$

$$= \theta^2$$

$E\left[\frac{T(T-1)}{n(n-1)}\right]$ is an unbiased estimator of θ^2

6. If T is an unbiased estimator of θ, such that T^2 is a biased estimator of θ^2.

Sol:

T is an unbiased estimator of θ

ie; $E(T) = \theta$

$V(T) = E(T^2) - [E(T)]^2$

$= E(T^2) - \theta^2$

$E(T^2) = V(T) + \theta^2$

$E(T^2) \neq \theta$.

T^2 is a biased estimator of θ^2

Also, $E(T) > 0$

$E(T^2) - \theta^2 > 0$

$E(T^2) > \theta^2$

$\boxed{\therefore \quad E(T^2) > \theta^2}$

7. If T is an unbiased estimator of θ. Verify whether $aT + b$ is an unbiased estimator of $a\theta + b$.

Proof:

T is an UBE of θ

$E(T) = \theta \rightarrow (1)$

$\Rightarrow E(aT+b) = aE(T) + E(b)$

$= a\theta + b$

$aT+b$ is an UBE of $a\theta+b$.

10. If $x_1, x_2, \dots x_n$ is a random sample of size 'n' from a normal population with mean 'μ' & variance 'σ^2' verify whether $s^2 = \frac{\Sigma(x_i-\bar{x})^2}{(n-1)}$ is an UBE of σ^2

Proof

Given, $s^2 = \frac{1}{n-1} \Sigma(x_i-\bar{x})^2$

$= \frac{1}{n-1} \Sigma(x_i-\bar{x})^2$ (Add & subtract μ)

$= \frac{1}{n-1} \Sigma(x_i-\mu+\mu-\bar{x})^2$

$= \frac{1}{n-1} \Sigma\{(x_i-\mu) - (\bar{x}-\mu)\}^2$

$= \frac{1}{n-1} \Sigma(x_i-\mu)^2 + \frac{1}{n-1} \Sigma(\bar{x}-\mu)^2 - \frac{2}{n-1} \Sigma(x_i-\mu)(\bar{x}-\mu)$

$$\boxed{\begin{aligned} \sigma^2 &= \frac{\Sigma(x_i-\mu)^2}{n} \\ n\sigma^2 &= \Sigma(x_i-\mu)^2 \end{aligned}}$$

$\frac{\Sigma x_i}{n} = \bar{x}$

$= \frac{n\sigma^2}{n-1} + \frac{1}{n-1} \Sigma(\bar{x}-\mu)^2 - \frac{2}{n-1}(n\bar{x}-n\mu)\Sigma(\bar{x}-\mu)$

$\Sigma\bar{x} - \Sigma\mu$

$$\boxed{\bar{x} = \Sigma x / n}$$

$\Sigma x = n\bar{x}$

$$\boxed{nx - n\cdot\mu}$$

$(n\bar{x} - n\mu)\cdot(n\bar{x} - n\mu)$

$$= \frac{n\sigma^2}{n-1} + \frac{1}{n-1}\Sigma(\bar{x}-\mu)^2 - \frac{2n}{n-1}(\bar{x}-\mu)(\bar{x}-\mu)$$

$$= \frac{n\sigma^2}{n-1} + \frac{1}{n-1}n(\bar{x}-\mu)^2 - \frac{2n}{n-1}(\bar{x}-\mu)^2$$

$$s^2 = \frac{n\sigma^2}{n-1} - \frac{n}{n-1}(\bar{x}-\mu)^2$$

Now,

$$E(s^2) = E\left[\left(\frac{n\sigma^2}{n-1}\right)\right] - \left[\frac{n}{n-1}E(\bar{x}-\mu)^2\right]$$

$$= \frac{n}{n-1}E(\sigma^2) - \frac{n}{n-1}\left[E(\bar{x}) - E(\mu)\right]^2$$

$$= \frac{n}{n-1}\cdot\sigma^2 - \frac{n}{n-1}\left[E(\bar{x}) - \bar{x}\right]^2$$

$$= \frac{n}{n-1}\cdot\sigma^2 - \frac{n}{n-1}V(\bar{x}) \qquad \therefore V(\bar{x}) = \sigma^2/n$$

$$= \frac{n}{n-1}\sigma^2 - \frac{\cancel{n}}{n-1}\cdot\frac{\sigma^2}{\cancel{n}} \Rightarrow \frac{n}{n-1}\left[\sigma^2 - \frac{\sigma^2}{n}\right]$$

$$\Rightarrow \frac{\cancel{n}}{\cancel{n}(n-1)}\left[n\sigma^2 - \sigma^2\right]$$

$$= \frac{1}{n-1}\sigma^2(n-1) = \sigma^2 \Rightarrow \frac{1}{\cancel{n-1}}\sigma^2\left[\cancel{n-1}\right]$$

$$\boxed{E(s^2) = \sigma^2}$$

s^2 is an unbiased estimator of σ^2

9. A random sample of size 'n' is taken from a normal population with mean 'μ' and variance 'σ^2', verify that s^2 is a biased estimator of σ^2 (15)

Proof:

Given $x \sim N(\mu, \sigma^2)$

$E(x) = \mu$, $V(x) = \sigma^2$ and

$\bar{x} \sim N(\mu, \sigma^2/n)$

$V(x) = E(x^2) - [E(x)]^2$

$\sigma^2 = E(x^2) - \mu^2$

$E(x^2) = \sigma^2 + \mu^2$ ——— (1)

If, $x \sim N(\mu, \sigma^2)$

$\bar{x} \sim N(\mu, \sigma^2/n)$

Then,

$$s^2 = \frac{1}{n} \Sigma (x_i - \bar{x})^2$$

$$= \frac{1}{n} \Sigma (x_i - \mu + \mu - \bar{x})^2$$

$$= \frac{1}{n} \Sigma \{(x_i - \mu) - (\bar{x} - \mu)\}^2$$

$$= 1/n\, \Sigma \{(x_i - \mu)^2 + (\bar{x} - \mu)^2 - 2(x_i - \mu)(\bar{x} - \mu)\}$$

$$= 1/n\, \Sigma (x_i - \mu)^2 + 1/n\, \Sigma (\bar{x} - \mu)^2 - \frac{2}{n} \Sigma (x_i - \mu)(\bar{x} - \mu)$$

$$= 1/n\, \Sigma (x_i - \mu)^2 + \frac{1}{n} \Sigma (\bar{x} - \mu)^2 - \frac{2n}{n}(\bar{x} - \mu)(\bar{x} - \mu)$$

$$s^2 = 1/n\, \Sigma (x_i - \mu)^2 + \frac{n}{n}(\bar{x} - \mu)^2 - 2(\bar{x} - \mu)^2$$

$$s^2 = \sigma^2 - (\bar{x}-\mu)^2$$

$$E(s^2) = E(\sigma^2) - E(\bar{x}-\mu)^2 \Rightarrow [E(\bar{x}^2) - E(\mu)]^2 \Rightarrow E(\bar{x}^2)$$

$$= \sigma^2 - V(\bar{x}) \Rightarrow \sigma^2 - (\sigma^2/n) \Rightarrow \sigma^2[1-1/n]$$

$$= \sigma^2 (1-1/n)$$

s^2 is a biased estimator of σ^2

10. Sample Variance is not an unbiased estimator of the population variance, but asymptotically unbiased.

proof:

To prove that sample variance is not an unbiased estimator that asymptotically unbiased.

Given : $x \sim N(\mu, \sigma^2)$

$$E(x) = \mu, \; V(x) = \sigma^2$$

And, $\bar{x} \sim N(\mu, \sigma^2/n)$

$$V(x) = E(x^2) - [E(x)]^2$$

$$\sigma^2 = E(x^2) - \mu^2$$

$$E(x^2) = \sigma^2 + \mu^2 \quad \text{——— } (1)$$

$$x \sim N(\mu, \sigma^2) ; \; \bar{x} \sim N(\mu, \sigma^2/n)$$

$$s^2 = 1/n \, \Sigma (x_i - \bar{x})^2$$

$$= 1/n \, \Sigma (x_i - \mu + \mu - \bar{x})^2$$

$\vdots$

$$s^2 = \sigma^2 - (\bar{x}-\mu)^2$$

$E(s^2) = \sigma^2(1-1/n) \Rightarrow \lim_{n\to\infty} E(s^2) = \lim_{n\to\infty} \sigma^2(1-1/n)$

$$\lim_{n\to\infty} E(s^2) = \sigma^2$$

$\therefore$ sample variance s^2 is asymptotically unbiased estimator of σ^2

Q. Such that the sample mean $\bar{x}$ is an unbiased estimator of $1/\theta$ for the distribution.

Proof:

Given; $f(x,\theta) = \theta(1-\theta)^{x-1}$, $x = 1, 2 \ldots \infty$

$$E(x) = \Sigma x \cdot p(x)$$

$$\sum_{1}^{\infty} x \cdot \theta(1-\theta)^{x-1}$$

$$= \theta\{1 + 2(1-\theta) + 3(1-\theta)^2 + \cdots\}$$

$$= \theta\{1-(1-\theta)\}^{-2}$$

$$= \theta\{(1-1+\theta)^{-2}\}$$

$$= \theta\{(\theta^{-2})\} \Rightarrow \frac{\theta}{\theta^2} = 1/\theta$$

$E(x) = 1/\theta$

$E(\bar{x}) = E(\Sigma x/n)$

$= 1/n \, E(\Sigma x)$

$= 1/n \, \Sigma E(x)$

$= \frac{1}{n} \cdot n \cdot \frac{1}{\theta}$

$$\boxed{E(\bar{x}) = 1/\theta}$$

$\therefore$ sample mean is an unbiased estimator of 1/

12. If T is an unbiased estimator of θ, then $\sqrt{T}$ is a biased estimator of $\sqrt{\theta}$ or if T is an UBE estimator of θ, then, T^2 is a biased estimator of θ^2 and also is not UBE $\sqrt{\theta}$.

Proof: Since $E(T^2) \neq \theta^2$,

$$V(\sqrt{T}) = E(\sqrt{T})^2 - [E(\sqrt{T})]^2$$

$\therefore T$ is an unbiased estimator of $\theta \Rightarrow E(T) = \theta$.

$$= \underline{E(T)} - [E(\sqrt{T})]^2$$

$$= \theta - [E(\sqrt{T})]^2$$

$\therefore V(\sqrt{T}) > 0$, we have

$$\theta - [E(\sqrt{T})]^2 > 0$$

$$\Rightarrow \theta > [E(\sqrt{T})]^2 \quad \text{(or)}$$

$$[E(\sqrt{T})]^2 < \theta$$

Taking Root on both sides, we get

$$E(\sqrt{T}) < \sqrt{\theta} \Rightarrow E(\sqrt{T}) \neq \sqrt{\theta}$$

$\therefore \sqrt{T}$ is a biased estimator of $\sqrt{\theta}$, T^2 is possitively biased estimator of θ^2, $\sqrt{T}$ is a negatively biased estimator of $\sqrt{\theta}$.

Consistancy Estimator :-

⇒ An estimator T_n which is based on sam[ple]
or n observations is called the Consistant estimator
of the parameter θ, if.

(i) $E(T_n) = \theta$

(ii) $V(T_n) = 0$ as $n \to \infty$

Sufficient condition for consistancy Estimator.

Let sequence $\{T_n\}$ be a sequence of estima-
-tors such that $E(T_n) = \theta$, then T is said to be
Consistant estimator for θ then,

$$V(T_n) \longrightarrow 0 \quad as\ n \to \infty$$

(1). Let $x_1, x_2, x_3, \dots x_n$ is a random sample of size n
is taken from a normal population with mean μ, var σ^2,
Verify whether $\bar{x}$, is a Consistant estimator of μ.

Proof:

Given, $x \sim N(\mu, \sigma^2)$

That in $E(x) = \mu$, $V(x) = \sigma^2$

then, $\bar{x} \sim N(\mu, \sigma^2/n)$

$\therefore E(\bar{x}) = \mu$ ——— ①

$V(\bar{x}) = \sigma^2/n$

$\lim_{n \to \alpha} V(\bar{x}) = \lim_{n \to \alpha} \sigma^2/n = 0$

$\therefore V(\bar{x}) = 0$, as $n \to \alpha$ ——— ②

From ① & ② we say the sample mean $\bar{x}$ is consis-taut estimator of μ.

$\bar{x}$ is consistaut estimator of μ.

2. Such that Sample median is Consistaut estimator for the population mean in normal population (or)

Let $x_1, x_2, \dots x_n$ be a random sample of size 'n' is form a normal population with mean μ & variance σ^2, whether the sample median is consistaut estimator of the population Mean.

Proof:

Given, $X \sim N(\mu, \sigma^2)$

that in $E(x) = \mu$, $V(x) = \sigma^2$

x - median $\sim N\left[\mu, \frac{\pi\sigma^2}{2n}\right]$

$\therefore E(x, \text{Median}) = \mu$ ——— ①

$V(x \text{ median}) = \frac{\pi\sigma^2}{2n}$

$\lim_{n \to \alpha} V(x \text{ median}) = \lim_{n \to \alpha} \frac{\pi\sigma^2}{2n} = 0$

$V(x \text{ median}) = 0$ as $n \longrightarrow \infty$ —— ②

∴ From ① & ② we say that sample median is consis-tant estimator of population mean μ.

③. Verify whether sample mean is a consistency estimator of the population mean when this sample is derived from a population which follows poisson Distribution.

Proof:

Let X be a random Variable follows poisson distribution with parameter λ that is $X \sim P(\lambda)$

$E(x) = \lambda \qquad V(x) = \lambda$

$$E(\bar{x}) = E\left[\frac{\Sigma x_i}{n}\right]$$

$$= \frac{1}{n} \Sigma E(x)$$

$$= \frac{1}{\cancel{n}} \cancel{n} \cdot \lambda$$

$$= \lambda \quad \text{—— ①}$$

$$V(\bar{x}) = V\left(\frac{\Sigma x}{n}\right)$$

$$= \frac{1}{n^2} V(\Sigma x)$$

$$= \frac{1}{n^2} \Sigma V(x)$$

$$= \frac{1}{n^{\cancel{2}}} \cancel{n} \cdot \lambda$$

$$= \lambda / n$$

$$\lim_{n\to\alpha} V(\bar{x}) = \lim_{n\to\alpha} x/n = 0 \quad \text{—— (2)}$$

From (1) & (2) equation. We day that sample mean is an Consistant estimator of population mean.

(4). A random sample of size 'n' is delected from a population which follows bernoulli distribution. Verify whether $\bar{x}$ is a Consistency estimator of parameter 'p' where P represents the probability of duccess.

Proof:

$X \sim$ Bernoulli distribution

$$P(x) = \begin{cases} P, & x = 1 \\ 1-P, & x = 0 \end{cases}$$

$$E(x) = \Sigma x_i \, p(x_i)$$

$$= 1 \times p + 0 \times (1-p)$$

$$= p + 0$$

$$\boxed{E(x) = p}$$

$$E(x^2) = \Sigma x^2 \cdot p(x)$$

$$= (1)^2 \times p + (0)^2 \times (1-p)$$

$$= 1 \times p = p$$

$$\boxed{E(x^2) = p}$$

$$V(x) = E(x^2) - [E(x)]^2$$

$$= p - p^2$$

$$= p(1-p)$$

$$E(\bar{x}) = E\left[\frac{\Sigma x_i}{n}\right]$$

$$= 1/n \; \Sigma (x/n)$$

$$= 1/n \cdot n(p)$$

$E(\bar{x}) = P$ —— ①

$V(\bar{x}) = v(\Sigma x/n)$

$\frac{1}{n^2} = \frac{1}{n^2} v(\Sigma x)$

$= \frac{1}{n^2} \Sigma v(x)$

$= \frac{1}{n^2} \cdot n \cdot P(1-P)$

$= \frac{1}{n} \cdot P(1-P)$

$\lim_{n \to \infty} v(\bar{x}) = 0$ —— ②

From 1 & 2 we say that sample mean $\bar{x}$ is cast. estimate of the parameter P

5. If random sample of size 'n' is selected from a normal population with mean μ and variance σ^2, verify whether S^2 is consistency estimator of σ^2

Proof:

$S^2 = \frac{1}{n-1} \Sigma (xi - \bar{x})^2$ $(+, \ominus = \mu)$

$= \frac{1}{n-1} \Sigma (xi - \mu + \mu - \bar{x})^2$

$= \frac{1}{n-1} \Sigma \{(xi - \mu) - (\bar{x} - \mu)\}^2$

$$= \frac{1}{n-1} \Sigma \left\{ (x-\mu)^2 + (\bar{x}-\mu)^2 - 2(x-\mu)(\bar{x}-\mu) \right\}$$

$$= \frac{1}{n-1} \Sigma (x_i-\mu)^2 + \frac{1}{n-1} \Sigma (\bar{x}-\mu)^2 - \frac{2}{n-1} \Sigma (\bar{x}-\mu)(\bar{x}-\mu)$$

$$s^2 = \frac{n \cdot \sigma^2}{n-1} + \frac{1}{n-1} \cdot n \cdot (\bar{x}-\mu)^2 - \frac{2}{n-1} \cdot n (\bar{x}-\mu)^2$$

$$s^2 = \frac{n\sigma^2}{n-1} - \frac{1}{n-1} \cdot n \cdot (\bar{x}-\mu)^2$$

$$E(s^2) = \frac{n}{n-1} E(\sigma^2) - \frac{n}{n-1} E(\bar{x}-\mu)^2$$

$$= \frac{n}{n-1} \sigma^2 - \frac{n}{n-1} E\left[\bar{x} - [E(\bar{x})]\right]^2$$

$$= \frac{n}{n-1} \sigma^2 - \frac{n}{n-1} V(\bar{x})$$

$$= \frac{n}{n-1} \sigma^2 - \frac{\cancel{n}}{n-1} \cdot \frac{\sigma^2}{\cancel{n}} \Rightarrow \frac{n}{n-1}\left[\sigma^2 - \frac{\sigma^2}{n}\right]$$

$$\Rightarrow \frac{\cancel{n}}{\cancel{n}(n-1)} [n\sigma^2 - \sigma^2) \Rightarrow \frac{\sigma^2}{\cancel{n-1}} \cancel{(n-1)}$$

$$= \frac{n\sigma^2}{n-1} - \frac{\sigma^2}{n-1} \Rightarrow \frac{\sigma^2 \cancel{(n-1)}}{\cancel{(n-1)}} = \sigma^2$$

$$E(s^2) = \sigma^2 \longrightarrow ①$$

$$\Rightarrow$$

$$V(s^2) = \frac{2\sigma^4}{n}$$

$$V(s^2) = \left(\frac{n}{n-1}\right)^2 \boxed{V(s^2)}$$

$$= \frac{n^2}{(n-1)^2} \cdot \frac{2}{n} \sigma^4$$

There $\boxed{n-1 \simeq n}$

$$= \frac{\cancel{n^2} \cdot 2}{\cancel{n^2} \cdot n} \sigma^4$$

$$V(s^2) = \frac{2\sigma^4}{n}$$

7. A random sample of size 'n' is defined from a normal population, whose density function $f(x) = \begin{cases} 1, & \theta < x < \theta+1 \\ 0, & \text{otherwise} \end{cases}$. Verify whether $\bar{x}$ is a consistancy estimator of $\frac{\theta+1}{2}$

Proof:

$f(x) = 1$ $\quad E x = \int x \cdot f(x) \cdot dx.$

$$E(x) = \int_{\theta}^{\theta+1} x \cdot 1 \cdot dx$$

$$= \left[\frac{x^2}{2}\right]_{\theta}^{\theta+1}$$

$$= \frac{1}{2}\left[(\theta+1)^2 - (\theta)^2\right]$$

$$= \frac{1}{2}\left[\cancel{\theta^2} + 1^2 + 2\theta - \cancel{\theta^2}\right]$$

$$= \frac{1}{2}\left[1+2\theta\right] \Rightarrow \frac{1}{2} + \frac{1}{2}2(\theta)$$

$$= \frac{1}{2} + \theta.$$

$$E(x^2) = \int_{\theta}^{\theta+1} x^2 \cdot 1 \cdot dx$$

$$= \left[\frac{x^3}{3}\right]_{\theta}^{\theta+1}$$

$$= \frac{1}{3}\left[(\theta+1)^3 - \theta^3\right] \qquad (a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$$

$$= \frac{1}{3}\left[\cancel{\theta^3} + 1^3 + 3\theta + 3\theta^2 - \cancel{\theta^3}\right]$$

$$= \frac{1}{3}\left[1 + 3\theta^2 + 3\theta\right]$$

$$= \frac{1}{3} + \theta^2 + \theta$$

7. A random sample of size 'n' is defined from a norm population, whose density function $f(x) = \begin{cases} 1, & 0 < x < \theta+1 \\ 0, & \text{otherwise} \end{cases}$

verify whether $\bar{x}$ is a consistancy estimator of $\theta + \frac{1}{2}$

Proof:

$f(x) = 1$ $\quad E x = \int x \cdot f(x) \cdot dx.$

$$E(x) = \int_0^{\theta+1} x \cdot 1 \cdot dx$$

$$= \left[\frac{x^2}{2}\right]_\theta^{\theta+1}$$

$$= \frac{1}{2}\left[(\theta+1)^2 - (\theta)^2\right]$$

$$= \frac{1}{2}\left[\cancel{\theta^2} + 1^2 + 2\theta - \cancel{\theta^2}\right]$$

$$= \frac{1}{2}\left[1 + 2\theta\right] \Rightarrow \frac{1}{2} + \frac{1}{\cancel{2}}\cancel{2}(\theta)$$

$$= \frac{1}{2} + \theta.$$

$$E(x^2) = \int_\theta^{\theta+1} x^2 \cdot 1 \cdot dx$$

$$= \left[\frac{x^3}{3}\right]_\theta^{\theta+1}$$

$$= \frac{1}{3}\left[(\theta+1)^3 - \theta^3\right] \qquad (a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$$

$$= \frac{1}{3}\left[\cancel{\theta^3} + 1^3 + 3\theta + 3\theta^2 - \cancel{\theta^3}\right]$$

$$= \frac{1}{3}\left[1 + 3\theta^2 + 3\theta\right]$$

$$= \frac{1}{3} + \theta^2 + \theta$$

$$V(x) = E(x^2) - [E(x)]^2$$

$$= \frac{1}{3} + \theta^2 + \theta - (\tfrac{1}{2} + \theta)^2 \qquad (\tfrac{1}{2}+\theta)^2 = \tfrac{1}{4} + \theta^2 + 2\cdot\tfrac{1}{2}\cdot\theta$$

$$= \frac{1}{3} + \cancel{\theta^2} + \cancel{\theta} - \tfrac{1}{4} - \cancel{\theta^2} - \cancel{\theta}^2$$

$$= \frac{1}{12}$$

$$E(\bar{x}) = E\left(\frac{\Sigma x}{n}\right) \Rightarrow \frac{1}{n}\,\Sigma\cdot E(x) \Rightarrow \frac{1}{n}\cdot n\cdot(\tfrac{1}{2}+\theta)$$

for Σ

$$= \tfrac{1}{2} + \theta \longrightarrow (1)$$

$$V(\bar{x}) = V\left(\frac{\Sigma x}{n}\right) \Rightarrow \frac{1}{n^2}\cdot V(\Sigma x)$$

$$= \frac{1}{n^2}\,\Sigma V(x) = \frac{1}{n^2}\cdot \cancel{n}\cdot\frac{1}{12} = \tfrac{1}{12}$$

$$\lim_{n\to\alpha} V(x) = 0, \quad \text{as } n \to \alpha \longrightarrow (2)$$

From (1) & (2), it is proved.

10. If $x_1, x_2, \dots x_n$ and random observation on Bernoulli variable, x taking the value '1' with probability 'p' and the value '0' with probability '1-p' such that $\frac{\Sigma x_i}{n}\left(1 - \frac{\Sigma x_i}{n}\right)$ is a Consistancy estimator of $p(1-p)$.

Proof:

$x_1, x_2 \dots x_n$ are independent identically distributed, Bernoulli variable with p.m.f then

$T = x_1 + x_2 + \cdots + x_n \rightarrow ①$ is a bernoulli variable

with parameters n & p

$$E(T) = np, \quad V(T) = npq$$

$$\bar{x} = \frac{\Sigma x_i}{n} = \frac{x_1 + x_2 + \cdots x_n}{n}$$

$$= \frac{T}{n}$$

$$E(\bar{x}) = E\left(\frac{T}{n}\right) \Rightarrow \frac{1}{n} E(T)$$

$$= \frac{1}{n} \cdot np$$

$$= p \longrightarrow ②$$

$$V(\bar{x}) = V\left(\frac{\Sigma x}{n}\right)$$

$$= V(T/n)$$

$$= 1/n^2 \cdot V(T)$$

$$= 1/n^{\cancel{2}} \cdot \cancel{n}pq$$

$$= \frac{pq}{n}$$

$$\lim_{n \to \infty} V(\bar{x}) = \lim_{n \to \infty} \frac{pq}{n} = 0 \longrightarrow ③$$

From ②, ③ $\bar{x}$ is a consistency Estimator of

parameter 'p'

Consistency :

* An estimator $T_n = T(x_1, x_2, \cdots x_n)$ based on random sample of size 'n' is said to be consistent estimator of $\gamma(\theta)$ where $\theta \in \Theta$ the parameter space if T_n converge to $\gamma(\theta)$ in probability.

$$\boxed{\text{if, } T_n \xrightarrow{P} \gamma(\theta) \text{ as } n \to \infty}$$

* In otherwords. T_n is a consistent estimator of $\gamma(\theta)$ if for every $\epsilon > 0$, $\eta > 0$ there exists. a positive integer $n \geq m$ such that

$$P\left(|T_n - \gamma(\theta)| < \epsilon\right) \to 1 \text{ as } n \to \infty$$

$$\Rightarrow P\left\{|T_n - \gamma(\theta)| < \epsilon\right\} > 1 - \eta \quad \forall\, n \geq m$$

where m is very large value of n.

Invariance property of consistent estimator :-

Theorem :- If T_n is a consistent estimator of $\gamma(\theta)$ and $\psi(\gamma(\theta))$ is a continuous function of $\gamma(\theta)$ Then $\psi(T_n)$ is a consistant estimator of $\psi(\gamma(\theta))$

Proof:

Since T_n is a consistent estimator of $\gamma(\theta)$

$\therefore T_n \xrightarrow{P} \gamma(\theta)$ as $n \to \infty$

for $\epsilon > 0$, $\eta > 0$ $\exists$ a positive integer $n \geq m$

(ϵ, η) such that,

$$P\{|T_n - \gamma(\theta)| < \epsilon\} > 1-\eta \quad \forall\ n \geq m$$

Since $\psi(\cdot)$ is a continuous function for every $\epsilon > 0$

however small $\exists$ a positive number ϵ_1 such that

$$\{|\psi(T_n) - \psi(\gamma(\theta))| < \epsilon_1\} \text{ whenever } |T_n - \gamma(\theta)| < \epsilon$$

$$\{|T_n - \gamma(\theta)| < \epsilon\} \Rightarrow \{|\psi(T_n) - \psi(\gamma(\theta))| < \epsilon_1\} \to ①$$

for Two event A and B

if, $A \Rightarrow B$ then $A \subseteq B$

$$P(A) \leq P(B) \text{ or } P(B) \geq P(A) \to ②$$

from ① & ② we get,

$$P[|\psi(T_n) - \psi(\gamma(\theta))| < \epsilon_1] \geq P[|T_n - \gamma(\theta)| < \epsilon]$$

$$\Rightarrow P[|\psi(T_n) - \psi(\gamma(\theta))| < \epsilon_1] \geq 1-\eta \quad \forall\ n \geq m$$

$$\therefore \psi(T_n) \xrightarrow{P} \psi(\gamma(\theta)) \text{ as } n \to \infty$$

or. $\varphi(T_n)$ is a consistent estimator of $\gamma(\theta)$ (33)

If $x_1, x_2 \ldots x_n$ are random observation on a bernoulli variate X taking the value 1 with probability p and the value 0 with probability $1-p$ show that

$$\frac{\Sigma x_i}{n}\left(1 - \frac{\Sigma x_i}{n}\right) \text{ is consistent estimator of } p(1-p)$$

Proof: $x_1, x_2 \ldots x_n$ are iid bernoulli variate with parameter P

$$\boxed{T = \sum_{i=1}^{n} x_i \sim B(n,p)}$$

$$E(T) = np$$

$$V(T) = npq$$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = T/n \Rightarrow \boxed{\bar{x} = T/n}$$

$$E(\bar{x}) = E(T/n) = \frac{1}{\cancel{n}} \cdot \cancel{n}p = P$$

$\Rightarrow E(T) \cdot 1/n$

$$E(\bar{x}) = P$$

$$Var(\bar{x}) = var(T/n) = 1/n^2 \cdot V(T) = \frac{1}{n^2} \cdot npq = \frac{pq}{n}$$

$$V(\bar{x}) = pq/n$$

$$V(T) = pq/n \longrightarrow 0 \text{ as } n \to \infty$$

since, $E(\bar{x}) \to P$ and $V(\bar{x}) \to 0$ as $n \to \infty$

$\bar{x}$ is consistent estimator of P

$\frac{\Sigma x_i}{n}\left(1-\frac{\Sigma x_i}{n}\right) = \bar{x}(1-\bar{x})$, being a polynomial

in $\bar{x}$, is a continuous function of $\bar{x}$

$\therefore$ $\bar{x}$ is consistency estimator of p, by the

invariance property of consistency estimator, $\bar{x}(1-\bar{x})$

is C.E of $p(1-p)$.