UNIT III

Mathematical Expectations

Expectations of a random variable

If $x_1, x_2 \ldots x_n$ are the discrete random variables with their Probabilities $P(x_1)\ P(x_2) \ldots P(x_n)$ then their mathematical expectation can be defined as

$$E(x) = \sum x_i P(x_i)$$

For the continuous random variable x and its probability density function is $f(x)$ then their expectation is defined as

$$E(x) = \int x\, f(x)\, dx$$

Expectations of mean, variance and moments

consider the random variable x and the mean is nothing but the expectation

$$\text{mean} = E(x)$$

$$\text{Variance} = E(x - E(x))^2$$

$$= E(x^2) - [E(x)]^2$$

moments :

The r^{th} raw moment is defined as

$$\mu_r' = E(x^r)$$

The r^{th} central moment is defined as

$$\mu_r = E[x - E(x)]^r$$

Addition theorem on expectations

Statement:

The mathematical expectation of the sum of random variables is equal to the sum of the expectations provided all the expectation exists

Symbolically If $x, y, z, \ldots T$ are n random variables then

$$E(x+y+z+\cdots+T) = E(x)+E(y)+E(z)+\cdots+E(T)$$

If all the expectation exists

Proof:

Let us consider two random variables x and y. The random variables x assume the values $x_1, x_2 \ldots x_m$ and their respective probabilities are $p_1, p_2 \ldots p_m$ where

$p_i = P[x = x_i]$ where $i = 1, 2, \ldots m$

The random variable y assume the values $y_1, y_2 \ldots y_n$ and their respective probabilities are $p_1', p_2', \ldots p_n'$ where

$p_j' = P[y = y_j]$ where $j = 1, 2, \ldots n$

By the definition of expectations

$$\left.\begin{aligned} E(x) &= \sum x_i p_i \\ E(y) &= \sum y_j p_j' \end{aligned}\right\} \quad \text{— (1)}$$

Since any one of the values of m values of x_i can be associated with any n values of y_j where $i = 1, 2 \ldots m$ and $j = 1, 2 \ldots n$

By def

$$E(x+y) = \sum_i \sum_j (x_i + y_j) p_{ij}$$

$$= \sum_i \sum_j x_i p_{ij} + \sum_i \sum_j y_j p_{ij}$$

$$= \sum_i x_i \sum_j p_{ij} + \sum_j y_j \sum_i p_{ij}$$

$$= \sum_i x_i p_i + \sum_j y_j p_j$$

where $p_i = \sum_{j=1}^{m} p_{ij}$; $p_j = \sum_{i=1}^{n} p_{ij}$

hence

$$E(X+Y) = E(X) + E(Y) \quad \text{—— (2)}$$

Now let $K = X+Y$.

$$E(K+Z) = E(K) + E(Z) \quad \text{from (2)}$$
$$= E(X+Y) + E(Z)$$
$$= E(X) + E(Y) + E(Z)$$

hence by mathematical induction

$$E(X+Y+Z+\cdots T) = E(X) + E(Y) + \cdots + E(Z)$$

Multiplication theorem on expectation

Statement:

The mathematical expectation of the product of no of independent random variable is equal to the product of their expectation. Symbolically if $X, Y, Z \ldots T$ are independent random variable then

$$E(X, Y, Z \ldots T) = E(X)\, E(Y) \ldots E(Z)$$

Proof:

Let us prove the theorem for two random variables X, Y

Let the random variables x assume the values $x_1, x_2 \dots x_m$ with their respective probabilities $p_1, p_2, \dots p_m$ where $p_i = p[x = x_i]$ $i = 1, 2 \dots m$

The random variable y assume the values $y_1 y_2 \dots y_n$ with their respective probabilities $p_1' \; p_2', \dots p_n'$ where

$$p_j' = p[y = y_j] \qquad j = 1, 2 \dots n$$

Then by the definition of expectation

$$\left.\begin{aligned} E(x) &= \Sigma x_i p_i \\ E(y) &= \Sigma y_j p_j' \end{aligned}\right\} \text{ --- } (1)$$

The product x, y is a random variable which can assume $m \times n$ values $x_i y_j$ $(i = 1, 2 \dots m, \; j = 1, 2 \dots n)$

$$p_{ij} = p[x = x_i] \cap p[y = y_j]$$
$$= p_i p_j' \qquad \text{as } x, y \text{ are independent}$$

By Def

$$E(xy) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j p_{ij}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j p_i p_j'$$

$$E(xy) = \sum_{i=1}^{m} x_i p_i \cdot \sum_{j=1}^{n} y_j p_j'$$

$$E(xy) = E(x) \, E(y) \text{ --- } (2)$$

consider the 3 r.v's x, y, z where $k = xy$

$$E(kz) = E(k) \cdot E(y) \qquad \text{from } (2)$$

$= E(xy)\ E(z)$

$E(xyz) = E(x)\,E(y)\,E(z)$

By the method of mathematical induction the result holds for $x, y, z \dots T$ also

So

$E(x, y, z \dots T) = E(x)\,E(y) \dots E(T)$

Addition theorem of Mathematical Expectation for continous r.v's

Statement:

If x and y are continous random variables with joint density function $f(x,y)$

Then

$E(x+y) = E(x) + E(y)$

Proof:

Given x and y are two random variables with joint probability density function $f(x,y)$

$$E(x+y) = \iint (x+y)\, f(x,y)\, dx, dy$$

$$= \int x \left[\int f(x,y)\,dy\right] dx + \int y \left[\int f(x,y)\,dx\right] dy$$

$$= \int x\, f(x)\, dx + \int y\, f(y)\, dy$$

as $f(x,y)\,dx = f(y)$

$f(x,y)\,dy = f(x)$

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$= E(x) + E(y)$

Multiplication theorem of Mathematical ~~expression~~ expectation for continous r.v's

Statement:

If x and y are continous random variables with joint density function f(x,y) Then

$E(xy) = E(x) \cdot E(y)$

Proof:

Given x and y are two independent random variables with joint density fn f(x,y) then

$$E(xy) = \iint xy\, f(x,y)\, dx\, dy$$

$$= \iint xy\, f(x)\, f(y)\, dx\, dy$$

since x and y are independent
$f(xy) = f(x)\, f(y)$

$$= \int x\, f(x)\, dx \int y\, f(y)\, dy$$

$$E(xy) = E(x)\, E(y)$$

Hence the proof.

Properties of Mathematical Expectation

1. If x is a r.v and a is constant then

i) $E[a\psi(x)] = a\,E[\psi(x)]$

ii) $E[\psi(x) + a] = E[\psi(x)] + a$

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where $\psi(x)$, a function of x is a random variable and all expectation exists

i) $E[a\psi(x)] = \int_{-\infty}^{\infty} a\,\psi(x) f(x)\,dx$

$$= a\int_{-\infty}^{\infty} \psi(x) f(x)\,dx$$

$$= a\,E[\psi(x)]$$

ii) $E[\psi(x)+a] = \int_{-\infty}^{\infty} [\psi(x)+a] f(x)\,dx$

$$= \int_{-\infty}^{\infty} \psi(x) f(x)\,dx + a\int_{-\infty}^{\infty} f(x)\,dx$$

$$= E[\psi(x)] + a \int_{-\infty}^{\infty} f(x)\,d$$

corollary: If $\psi(x) = x$ then

$E[ax] = a\,E(x)$ and
$E[x+a] = E(x)+a$

2. If x is a r.v's and a and b are constant then

$$E[ax+b] = a\,E(x)+b.$$

provided all the expectation exists

Proof:

By def

$$E[ax+b] = \int_{-\infty}^{\infty} (ax+b) f(x)\,dx$$

$$= a\int_{-\infty}^{\infty} x\, f(x)\,dx + b\int_{-\infty}^{\infty} f(x)\,dx$$

$= a\,E(x) + b(1)$ as $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$= a\,E(x)$

Properties of Variance.

1. If x is a r.v then $V(ax+b) = a^2 V(x)$ where a and b are constants

Proof: Let $y = ax+b$
Then

$E(y) = aE(x) + b$.

subtracting the above two

$y - E(y) = ax + \cancel{b} - aE(x) - \cancel{b}$

$= a[x - E(x)]$

squaring and taking expectations on Both sides

$E[y - E(y)]^2 = a^2 E[x - E(x)]^2$

$\Rightarrow V(y) = a^2 V(x)$

or

$V(ax+b) = a^2 V(x)$

Covariance.

If x and y are two r.v's then co-variance between them is defined as

$cov(xy) = E\{[x - E(x)][y - E(y)]\}$

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$$\text{cov}(x,y) = E[(x-E(x))(y-E(y))]$$

$$= E[xy - xE(y) - yE(x) + E(x)E(y)]$$

$$= E[xy) - E(y)E(x) - \cancel{E(x)E(y)} + \cancel{E(x)E(y)}$$

$$= E(xy) - E(x)E(y)$$

$\therefore$

$$\text{cov}(x,y) = E(xy) - E(x)E(y)$$

Expectation of a liniar combination of

Statement:

Let $x_1, x_2 \ldots x_n$ be n random variables and if $a_1, a_2, \ldots a_n$ be any constants then

$$E\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i E(x_i)$$

Proof: We know that

The mathematical expectation of sum of n random variables is equal to the sum of their expectation

ie

$$E(x_1 + x_2 + \cdots + x_n) = E(x_1) + E(x_2) + \cdots + E(x_n)$$

also $E(ax) = aE(x)$

The theorem is proved by the principle of mathematical induction for

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Let us assume the theorem is true for $n = k$

$y = a_1 x_1 + a_2 x_2 + \cdots + a_k x_k$.

consider 2 variables x_1 and x_2 and constants a_1, a_2

$y = a_1 x_1 + a_2 x_2$

$E(y) = E[a_1 x_1 + a_2 x_2)$

$= E[a_1 x_1] + E(a_2 x_2]$

$= a_1 E(x_1) + a_2 E(x_2)$ it is true for $n = 2$

now suppose it is true for $n = k$

$y = a_1 x_1 + a_2 x_2 + \cdots + a_k x_k$

Taking expectation

$E(y) = a_1 E(x_1) + a_2 E(x_2) + \cdots + a_k E(x_k)$

Then let $n = k+1$

$y = a_1 x_1 + a_2 x_2 + \cdots + a_k x_k + a_{k+1} x_{k+1}$

Taking expectation

$E(y) = a_1 E(x_1) + a_2 E(x_2) + \cdots + a_{k+1} E(x_{k+1})$

The theorem is true for $n = 2$, $n = k$, $n = k+1$ hence it is true for positive values of n

$\therefore$

$$E\left(\sum_{i=1}^{n} a x_i\right) = \sum_{i=1}^{n} a_i E(x_i)$$

Variance of a linear combination of R.V's.

Statement:

Let $x_1, x_2 \dots x_n$ be n r.v's

$$V\left[\sum_{i=1}^{n} a_i x_i\right] = \sum_{i=1}^{n} a_i^2 V(x_i) + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \operatorname{cov}(x_i x_j)$$

Proof:

Let $U = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$

Take expectation

$$E(U) = a_1 E(x_1) + a_2 E(x_2) + \cdots + a_n x_n$$

subtracting the above

$$U - E(U) = a_1 (x_1 - E(x_1)) + a_2 (x_2 - E(x_2)) + \cdots + \cdots + a_n (x_n - E(x_n))$$

squaring the above and taking expectation

$$E[U - E(U)]^2 = E[a_1(x_1 - E(x_1)) + a_2 (x_2 - E(x_2)) + \cdots + a_n (x_n - E(x_n))]^2$$

$$= a_1^2 E(x_1 - E(x_1))^2 + a_2^2 E(x_2 - E(x_2))^2 + \cdots + a_n^2 E(x_n - E(x_n))^2 + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j E[x_i - E(x_i)][x_j - E(x_j)]$$

$$\Rightarrow V(U) = a_1^2 V(x_1) + a_2^2 V(x_2) + \cdots a_n^2 V(x_n) + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \operatorname{cov}(x_i x_j)$$

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$$V\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i^2 V(x_i) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \operatorname{cov}(x_i x_j)$$

Remarks

1. If $a_i = 1$ $\quad i = 1, 2 \ldots n$

$$V(x_1 + x_2 + \cdots + x_n) = V(x_1) + V(x_2) + \cdots + V(x_n) + 2\sum_{\substack{i=1 \\ i<j}}^{n}\sum_{j=1}^{n} \operatorname{cov}(x_i x_j)$$

2. If x and y are independent

$$V(x_1 + x_2 \cdots + x_n) = V(x_1) + V(x_2) + \cdots + V(x_n)$$

3. If c is a constant.

$$E(c) = \int_{-\infty}^{\infty} c\, f(x)\, dx$$

$$= c\int_{-\infty}^{\infty} f(x)\, dx$$

$$= c$$

$\therefore\ E(c) = c$

4 If x and y are 2 r.vs such that $y \leq x$ then

$$E(y) \leq E(x)$$

Proof:

Since $y \leq x \Rightarrow y - x \leq 0$

or $x - y \geq 0$

$$E(x-y) \geq 0 \Rightarrow E(x) - E(y) \geq 0$$

$$E(x) \geq E(y)$$

$$\Rightarrow E(y) \leq E(y)$$

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Examples

1. Let X be discrete r.v with the following p.m.f

X :	-2	-1	0	1	2	3
$P(x)$:	0.1	k	0.2	$2k$	0.3	k

i) Determine k.

ii) Find mean and variance

i) $\Sigma P_i = 1$

$$\Rightarrow 0.1 + k + 0.2 + 2k + 0.3 + k = 1$$

$$4k + 0.6 = 1$$

$$4k = 1 - 0.6$$

$$k = 0.4/4$$

$$k = 0.1$$

Now

X :	-2	-1	0	1	2	3
$P(X)$:	0.1	0.1	0.2	0.2	0.3	0.1

Mean : $E(x)$

$$E(X) = \Sigma x P(x)$$

$$= (-2)(0.1) + (-1)(0.1) + 0 + 1(0.2) + (2)(0.3) + 3(0.1)$$

$$= \cancel{-0.2} - 0.1 + \cancel{0.2} + 0.6 + 0.3$$

$$= -0.1 + 0.9$$

$$= 0.8$$

<u>Variance</u> :

$$V(X) = E(X^2) - E(X)^2$$

X^2 :	4	1	0	1	4	9
$P(X)$:	0.1	0.1	0.2	0.2	0.3	0.1

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$E(x^2) = \Sigma x^2 P(x)$

$= (4)(0.1) + (1 \times 0.1) + (0 \times 0.2) + (1 \times 0.2)$
$+ (4 \times 0.3) + (9 \times 0.1)$

$= 2.8$

$V(x) = E(x^2) - E(x)^2$
$= 2.8 - (0.8)^2$
$= 2.8 - 0.64$
$= 2.16$

2. Given the following data

x :	-3	-2	-1	0	1	2	3
$P(x)$	0.05	0.10	0.30	0	0.30	0.15	0.10

compute

i) $E(x)$ ii) $E(2x \pm 3)$ iii) $E(4x+3)$
iv) $E(x^2)$ v) $V(x)$ vi) $V(2x \pm 3)$

$E(x) = (-3 \times 0.05) + (-2 \times 0.10) + (-1 \times 0.30) + 0$
$+ (1 \times 0.30) + (2 \times 0.15) + (3 \times 0.10)$

$= -0.15 - 0.20 - 0.30 + 0.30 + 0.3 + 0.3$
$= 0.25$

$E(2x \pm 3) = E(2x+3)\ E(2x-3)$

$E(2x+3) = 2E(x) + 3$
$= 2(0.25 + 3)$
$= 3.5$

$E(2x-3) = 2.E(x) - 3$
$= 2(0.25) - 3$
$= 0.5 - 3 = \cancel{0.28}$
$= -2.5$

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$E(2x \pm 3) = 3.5, -2.5$

iii) $E(4x+5) = 4.E(x) + 5$

$= 4 \times 0.25 + 5$

$= 6$

iv) $E(x^2) = 9 \times 0.05 + 4 \times 0.10 + 1 \times 0.30 + 0 + 1 \times 0.30$

$+ 4 \times 0.15 + 9 \times 0.10$

$= 2.95$

v) $V(x) = E(x^2) - E(x)^2$

$= 2.95 - (0.25)^2$

$= 2.95 - 0.0625$

$= 2.8875$

vi) $V(2x \pm 3) = V(2x+3) \,.\, V(2x-3)$

$V(2x+3) = 2^2 V(x) + 3$

$= 4 \times 2.8875 + 3$

$= 11.55 + 3$

$= 14.55$

$V(2x-3) = 2^2 V(x) - 3$

$= 4 \times 775 - 3$

$= 11.55 - 3$

$= 8.55$

vii) $\therefore V(2x \pm 3) = 8.55, 14.55$

3. Let x be r.v with the foll p.distn

X :	-3	6	9
P(X=x)	1/6	1/2	1/3

Find $E(x)$ $E(x^2)$ $E(2x+1)^2$ $V(x)$

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4 For the foll p.d.f find $E(x)$ and $V(x)$

i) $f(x) = \frac{1}{2}(x+1)$ $\quad -1 \le x \le 1$

ii) $f(x) = y_0(x-x^2)$ $\quad 0 \le x \le 1$

i) $f(x) = \frac{1}{2}(x+1)$

$$E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{-1}^{1} \frac{1}{2}(x+1)\, dx$$

$$= \frac{1}{2}\int_{-1}^{1}\left[(x+1)\, dx\right]$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + x\right]_{-1}^{1}$$

$$= \frac{1}{2}\left\{\left(\frac{1}{2}+1\right) - \left(\frac{1}{2}-1\right)\right\}$$

$$= \frac{1}{2}\left[\frac{1}{2}+1 - \frac{1}{2}+1\right]$$

$$= \frac{1}{\cancel{2}} \times \cancel{2}$$

$$= 1$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} x^2(x+1)\, dx$$

$$= \frac{1}{2}\int_{-1}^{1}(x^3+x^2)\, dx$$

$$= \frac{1}{2}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{1}{4}+\frac{1}{3}-\left(\frac{1}{4}-\frac{1}{3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{3}+\frac{1}{3}\right]$$

$$= 1/3$$

$$V(X) = E(X^2) - E(X)^2$$

$$= \frac{1}{3} - (1)^2$$

$$= \frac{1}{3} - 1$$

$$= -2/3$$