Unit - III

CONVERGENT AND DIVERGENT SERIES

Definition :- [Infinite series]

The infinite series $\sum_{n=1}^{\infty} a_n$ is an ordered pair $< \{a_n\}_{n=1}^{\infty}, \{s_n\}_{n=1}^{\infty} >$ where $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers; and

$s_n = a_1 + a_2 + \dots\dots + a_n$

the number a_n is called the n^{th} term of the series. The number s_n is called the n^{th} partial sum of the sums.

$\{1, 2, 3, \dots\dots\dots\} \rightarrow$ sequence

$\{a_1, a_2, a_3 \dots\dots\dots\}$

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3$

$\dots\dots\dots$ $s_n = a_1 + a_2 + \dots + a_n$

$\{s_1, s_2, s_3 \dots\dots, s_n\}$

Example :-

Consider the series

$1 + x + x^2 + \dots\dots + x^n + \dots\dots$

Thus can be written as $\sum_{n=0}^{\infty} x^n$

$a_0 = 1, a_1 = x, a_2 = x^2, \dots\dots$

$s_0 = 1, \quad s_1 = a_0 + a_1 = 1 + x$

$s_2 = a_0 + a_1 + a_2 = 1 + x + x^2$

$s_n = a_0 + a_1 + a_2 + \dots\dots + a_n$

$\Rightarrow 1 + x + x^2 + \dots\dots + x^n$

The convergence and divergence of the series depend's on the Convergence and divergence of the sequence $\{S_n\}_{n=1}^{\infty}$ of partial sums.

Definition :- [Convergent / Divergnent series]

Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers with partial Sums $S_n = a_1 + a_2 + \ldots\ldots + a_n$ $(n \in J)$

If the sequence $\{S_n\}_{n=1}^{\infty}$ Converges to $A \in R$ we say that thed series $\sum_{n=1}^{\infty}$ an covergent

If $\{S_n\}_{n=1}^{\infty}$ diverges we say that $\sum_{n=1} a_n$ diverges.

Thearem :-

If $\sum_{n=1}^{\infty} a_n$ convergent to A and $\sum_{n=1}^{\infty} b_n$ Convergent B then $\sum_{n=1}^{\infty} (a_n + b_n)$ convergent to $A+B$

Also if $C \in R$, then $\sum_{n=1}^{\infty} c\, a_n$ convergent to CA

Proof :-

If $S_n = a_1 + a_2 + \ldots\ldots + a_n$

and $t_n = b_1 + b_2 + \ldots\ldots + b_n$

Thus $\{S_n\}_{n=1}^{\infty}$ convergent to A and $\{t_n\}_{n=1}^{\infty}$ convergent is B,

Therefore by convergence sequences operation,

$\{S_n + t_n\}_{n=1}^{\infty}$ Converges to $A+B$

When $S_n + t_n = a_1 + a_2 + \ldots\ldots + a_n + b_1 + b_2 + \ldots\ldots + b_n$

$= (a_1 + b_1) + (a_2 + b_2) + \ldots + (a_n + b_n)$

Thus $\sum_{n=1}^{\infty} (a_n + b_n)$ covergens to $A+B$

Also, $cs_n = ca_1 + ca_2 + ca_3 + \dots + ca_n$

$= c(a_1 + a_2 + a_3 + \dots + a_n)$ whose

$\{cs_n\}$ converges to cA

$\therefore \sum_{n=1}^{\infty} c(a_n)$ Converges to CA.

$\therefore$ proved//

Thearem :-

If $\sum_{n=1}^{\infty} a_n$ is a convergent series then

$\lim_{n\to\infty} a_n = 0$

Proof :-

Suppose $\sum_{n=1}^{\infty} a_n = A$. Then

$\lim_{n\to\infty} s_n = A$ Where $s_n = a_1 + a_2 + \dots + a_n$

But, $\lim_{n\to\infty} s_{n-1} = A$ since $a_n = s_n - s_{n-1}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1}$

$= A - A$

$= 0$

$s_n = (a_1 + a_2 + \quad) + a_n$

$s_{n-1} = a_1 + a_2 + \dots$

$s_n = s_{n-1} + a_n$

$s_n - s_{n-1} = a_n$

$\therefore$ proved//

Promblem's :-

1) P.T the series $\sum_{n=1}^{\infty} \frac{(1-n)}{(1+2n)}$ must diverge.

To prove the series $\sum_{n=1}^{\infty} \frac{1-n}{1+2n}$ diverges it is enough to prove $\lim_{n\to\infty} a_n \neq 0$

here $a_n = \frac{1-n}{1+2n}$ (1)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1-n}{1+2n} = \lim_{n\to\infty} \frac{n\left(\frac{1}{n}-1\right)}{n\left(\frac{1}{n}+2\right)}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}-1}{\frac{1}{n}+2} = \frac{0-1}{0+2}$$

$$= \frac{-1}{2} \neq 0$$

$\therefore \sum_{n=1}^{\infty} \frac{1-n}{1+2n}$ diverges.

2) Check whether $\sum_{n=1}^{\infty} \frac{n+1}{n+2}$ is convergent or divergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n+2} = \lim_{n\to\infty} \frac{n(1+1/n)}{n(1+2/n)}$$

$$\lim_{n\to\infty} \frac{n+1/n}{1+2/n} = 1 \neq 0$$

$\therefore \sum_{n=1}^{\infty} a_n$ is divergent.

NOTE :-

If $\lim_{n\to\infty} a_n = 0$ then we cannot say $\sum a_n$ is convergent.

Series with nonnegative term's :-

Thearom :-

If $\sum_{n=1}^{\infty} a_n$ is a series of non-negative terms with $S_n = a_1 + a_2 + \cdots + a_n$ $(n \in I)$, then **(a)** $\sum_{n=1}^{n} a_n$ Converges if the sequence $\{S_n\}_{n=1}^{\infty}$ is bounded.
(b) $\sum_{n=1}^{\infty} a_n$ diverges if $\{S_n\}_{n=1}^{\infty}$ is not bounded.

Proof :-

a) Since $a_{n+1} \geq 0$

$$S_{n+1} = a_1 + a_2 + \cdots + a_n + a_{n+1}$$
$$= S_n + a_{n+1} \geq S_n$$
$$\therefore S_{n+1} \geq S_n \Rightarrow \{S_n\}_{n=1}^{\infty}$$ is a nondecreasing sequence

Thus $\{S_n\}_{n=1}^{\infty}$ is bounded and nondecreasing sequence.

Therefore $\{S_n\}_{n=1}^{\infty}$ is convergent.

$\Rightarrow \sum_{n=1}^{\infty} a_n$ is a Convergent series.

b) If $\{S_n\}_{n=1}^{\infty}$ is not bounded then $\{S_n\}_{n=1}^{\infty}$ divergent (ie) $\sum_{n=1}^{\infty} a_n$ divergent.

Thearom :-

a) If $0 < x < 1$, then $\sum_{n=1}^{\infty} x^n$ converges to $\frac{1}{1-x}$ (b) If $x \geq 1$ then $\sum_{n=1}^{\infty} x^n$ diverges.

Proof :-

(b) Since $x \geq 1$, $\{x^n\}_{n=1}^{\infty}$ does not converge to Zero $\therefore$ divergent.

Thus $\sum_{n=1}^{\infty} x^n$ is divergent.

a) Here $0 < x < 1$

$S_n = 1 + x + x^2 + \dots + x^n$

$= \frac{1 - x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$

If $0 < x < 1$ then. $\lim_{n \to \infty} x^{n+1} = 0$

$\therefore \lim_{n \to \infty} \frac{x^{n+1}}{1-x} = 0$

Thus $\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1}{1-x} - \lim_{n \to \infty} \frac{x^{n+1}}{1-x}$

$= \frac{1}{1-x} - 0$

$= \frac{1}{1-x}$

Thus $\{S_n\}_{n=0}^{\infty}$ converges to $\frac{1}{1-x}$

(ie) $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$

Theorem:

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

Proof:

Let the sequence be,

$\{S_n\}_{n=1}^{\infty}$ where $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$

Consider the subsequence $\{S_{2^n}\}_{n=1}^{\infty}$

(ie) $S_2, S_4, S_8, S_{16}, S_{32}, \dots$

$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 2$

$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = S_4 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$

$> S_4 + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$

$= 2 + \frac{1}{2} = 5/2$

$(S_{2^3})\ S_8 > 5/2$

Thus $S_{2^n} > \frac{n+2}{2}$

The Sequence $\{S_{2^n}\}_{n=1}^{\infty}$ is divergent

Since its $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+2}{2} \neq 0$

Since the subsequence $\{S_{2^n}\}$ is divergent.

$\{S_n\}_{n=1}^{\infty}$ is divergent.

harmonic Series divergent ?

NOTE :-

1) Sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ is Convergent, here $S_1 = \frac{1}{1}$, $S_2 = \frac{1}{2}$, $S_3 = \frac{1}{3}$ ...

2) Series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, here $S_1 = 1$, $S_2 = 1 + \frac{1}{2}$,

$S_3 = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

3) This series $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ is known as harmonic series.

Theorem :-

If $\sum_{n=1}^{\infty} a_n$ is a divergent series of positive number's. Then there is a sequence $\{\epsilon_n\}_{n=1}^{\infty}$ of positive numbers which convergent to zero but for which $\sum_{n=1}^{\infty} \epsilon_n a_n$ still diverges.

Proof :

Let $S_n = a_1 + a_2 + \cdots + a_n$

first let us show that $\sum_{n=1}^{\infty} \frac{S_{k+1} - S_k}{S_{k+1}}$ diverges

For any $m \in I$ choose $n \in I$ such that $S_{n+1} > 2S_n$

(since $\{S_k\}_{k=1}^{\infty}$ diverges to infinity).

Now $\{S_k\}_{k=1}^{\infty}$ is nondecreasing.

if $k \le n$

$S_{k+1} \le S_{n+1}$

$\frac{1}{S_{k+1}} \ge \frac{1}{S_{n+1}}$

Hence, $\sum_{k=m}^{n} \frac{S_{k+1} - S_k}{S_{k+1}} \ge \sum_{k=m}^{n} \frac{S_{k+1} - S_k}{S_{n+1}}$

$\Rightarrow \cancel{\frac{1}{S_{n+1}} \left(S_{m+1} - S_m + S_{m+2} - S_{m+1} + S_{m+3} - S_{m+2} + \dots + S_{n+1} - S \right.}$

$\Rightarrow \frac{1}{S_{n+1}} \left(S_{m+1} - S_m + S_{m+2} - S_{m+1} + S_{m+3} - S_{m+2} + \dots + S_{n+1} - S_n \right)$

$= \frac{S_{n+1} - S_m}{S_{n+1}}$

$\ge \frac{S_{n+1} - \frac{1}{2} S_{n+1}}{S_{n+1}}$ [since $\frac{1}{2} S_{n+1} > S_m$

$-\frac{1}{2} S_{n+1} < -S_m$

$-S_m > -\frac{1}{2} S_{n+1}$)

$= 1 - \frac{1}{2}$

$= \frac{1}{2}$

Thus for any $m \in I$ there exists $n \in I$ such that

$$\sum_{k=m}^{n} \frac{S_{k+1} - S_k}{S_{k+1}} \ge \frac{1}{2}$$

Since the partial sum of the series $\sum_{k=1}^{\infty} \frac{S_{k+1} - S_k}{S_{k+1}}$ is greater than (or) equal to $\frac{1}{2}$

$S_{k+1} = 1 + a_1 + a_2 + \dots + a_k + a_{k+1}$

$S_k = 1 + a_1 + a_2 + \dots + a_k$

Thus $\sum_{k=1}^{\infty} \frac{S_{k+1} - S_k}{S_{k+1}} = \infty$

But $S_{k+1} - S_k = a_{k+1}$

$\therefore \sum_{n=1}^{\infty} \frac{a_{k+1}}{S_{k+1}} = \sum_{k=2}^{\infty} \frac{a_k}{S_k} = \infty$

Let $\epsilon_k = \frac{1}{S_k}$ Then $\epsilon_k \to 0$ as $k \to \infty$

and $\sum_{k=2}^{\infty} a_k \epsilon_k = \infty$

$\therefore$ Proved //

Alternating Series :-

An Alternating Series is a series with terms alternate in sign. The Alternating series may be written as

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n \cdot \text{ (or)}$$

$$\sum_{n=1}^{\infty} (-1)^n a_n \text{ (With first term as negative)}$$

Theorem :-

If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that A) $a_1 \geq a_2 \geq a_3 \geq \dots \dots \geq a_n \geq a_{n+1} \geq \dots$

((ie) nonincreasing) and

and b) . $\lim_{n \to \infty} a_n = 0$ then the alternating Series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is convergent.

Proof:

Consider the partial sum with odd index.

$s_1, s_3, s_5, \dots \dots$ we have

$$s_3 = a_1 + a_2 + a_3$$

$$= s_2 + a_3$$

$$= s_1 - a_2 + a_3$$

by (a) $a_1 \geq a_2 \geq a_3 \geq \dots$

this implies $a_3 \leq a_1$

(ie) $s_3 \leq s_1$

For any $n \in I$ we have,

$$s_{2n+1} = s_{2n-1} - a_{2n} + a_{2n+1} \leq s_{2n-1}$$

Thus, $s_1 \geq s_3 \geq \dots \dots \geq s_{2n-1} \geq s_{2n+1} \geq \dots$ So that

$\{s_{2n+1}\}_{n=1}^{\infty}$ is nonincreasing.

But $s_{2n-1} = a_1 - a_2 + a_3 - a_4 + \dots \dots + a_{2n-3} - a_{2n-2} + a_{2n-1}$

Since, $a_{2n-1} > 0$, we have $s_{2n-1} > 0$

Hence, $\{s_{2n-1}\}_{n=1}^{\infty}$ is convergent

Similarly the sequence $s_2, s_4, s_6, \dots \dots, s_{2n}, \dots$ is convergent.

For $s_{2n+2} = s_{2n} + a_{2n+1} - a_{2n+2} \geq s_{2n}$

Therefore $\{s_{2n}\}_{n=1}^{\infty}$ is nondecreasing.

Also $s_{2n} = a_1 - (a_2 - a_3) - \dots \dots - (a_{2n-2} - a_{2n-1}) - a_{2n}$

$\leq a_1$

Therefore $s_{2n} \leq a_1$ so that $\{s_{2n}\}_{n=1}^{\infty}$ is bounded above.

Let $M = \lim_{n\to\infty} s_{2n-1}$ and $L = \lim_{n\to\infty} s_n$

Then $a_{2n} = s_{2n} - s_{2n-1}$

$s_{2n} = a_1 - a_2 + \dots + a_{2n}$

$s_{2n-1} = a_1 - a_2 + \dots + a_{2n-1}$

Given that $\lim_{n\to\infty} a_n = 0$

$\therefore \lim_{n\to\infty} a_{2n} = 0$

$\lim_{n\to\infty} (s_{2n} - s_{2n-1}) = 0$

$L - M = 0$

Thus $L = M$

Therefore both sequences $\{s_n\}_{n=1}^{\infty}$ and $\{s_{2n-1}\}_{n=1}^{\infty}$ convergent to L.

$\{s_{2n}\}_{n=1}^{\infty}$ and $\{s_{2n-1}\}_{n=1}^{\infty}$ Converges to L

Thus $\{s_n\}_{n=1}^{\infty}$ Converges to L

$\therefore$ proved //

Conditional Convergence and absolute Convergence

Definition :-

Let $\sum_{n=1}^{\infty} a_n$ be a series of real nnumber

(a) If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(b) If $\sum_{n=1}^{\infty} a_n$ Converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, we say that $\sum_{n=1}^{\infty} a_n$ Converges Conditionally.

example's :-

$\Rightarrow$ Consider the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots\cdots$

Which is a convergent series

$\Rightarrow$ But $\sum_{n=1}^{\infty} |a_n| = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\cdots$ diverges Convergent

$\Rightarrow$ Thus, $\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots\cdots$

Converges Conditionally. absolutely

2) Consider the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots$

$\Rightarrow \sum_{n=1}^{\infty} |a_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots$ diverges

$\therefore \sum_{n=1}^{\infty} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots$ Converges Conditionally.

Theorem :-

If $\sum_{n=1}^{\infty} a_n$ Converges absolutely, then $\sum_{n=1}^{\infty} a_n$ Converges

Proof :

Let $s_n = a_1 + a_2 + \ldots\ldots + a_n$

We have to prove that $\{s_n\}_{n=1}^{\infty}$ is Convergent

Since $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

$\sum_{n=1}^{\infty} |a_n| < \infty$ thus $\{t_n\}_{n=1}^{\infty}$ Converges

Where $t_n = |a_1| + |a_2| + \ldots\ldots + |a_n|$

Thus $\{t_n\}_{n=1}^{\infty}$ is Cauchy sequence.

Thus given $\epsilon > 0$. there exists $N \in I$ such that

$|t_m - t_n| < \epsilon \quad (m, n \geq N)$

But if $m > n$,

$|s_m - s_n| = |a_{n+1} + \ldots\ldots + a_m|$

$\leq |a_{n+1}| + |a_{n+2}| + \ldots + |a_m|$

$= |t_m - t_n|$

$\therefore |s_m - s_n| < \epsilon \quad (m, n \geq N)$

Thus $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence

$\therefore \{s_n\}_{n=1}^{\infty}$ Converges.

$\therefore \sum_{n=1}^{\infty} a_n$ is Convergent Series.

Thearem:

(a) If $\sum_{n=1}^{\infty} a_n$ Converges absolutely then both $\sum_{n=1}^{\infty} P_n$ and $\sum_{n=1}^{\infty} q_n$ converge (Where $a_n = P_n + q_n$

$P_n = Max(a_n, 0)$ (+ve terms)

$q_n = Min(a_n, 0)$ (-ve terms)

however,

(b) If $\sum_{n=1}^{\infty} a_n$ Converges conditionally, then both $\sum_{n=1}^{\infty} P_n$ and $\sum_{n=1}^{\infty} q_n$ diverge.

Proof:

(a) If $\sum_{n=1}^{\infty} a_n$ Converges absolutely then $\sum_{n=1}^{\infty} |a_n|$ Converges and by previous theorem $\sum_{n=1}^{\infty} a_n$ also converges.

Thus $\sum_{n=1}^{\infty} (a_n + |a_n|)$

Converges,

Since $P_n = Max(a_n, 0)$

$a_n + |a_n| = 2P_n$

Thus $\sum_{n=1}^{\infty} 2P_n$ Converges

Thus $\sum_{n=1}^{\infty} P_n$ Converges

Also $q_n = Min(a_n, 0)$

$2q_n = a_n - |a_n|$

Since $\sum_{n=1}^{\infty} (a_n - |a_n|)$ converges,

$\sum_{n=1}^{\infty} 2q_n$ converges

Thus, $\sum_{n=1}^{\infty} q_n$ converges

$\therefore \sum_{n=1}^{\infty} p_n$ & $\sum_{n=1}^{\infty} q_n$ converges.

(b) $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

$\therefore \sum_{n=1}^{\infty} |a_n|$ is divergent

$2p_n = a_n + |a_n|$

Since $\sum p_n$ is bounded & $\sum_{n=1}^{\infty} |a_n|$ is divergent $\sum p_n$ is divergent.

IIIy $2q_n = a_n - |a_n|$

Since $|a_n|$ diverges

$\sum_{n=1}^{\infty} 2q_n$ diverges

$\sum_{n=1}^{\infty} q_n$ diverges

$\therefore$ proved//

Rearrangement of Series :

Definition :

Let $N = \{n_i\}_{i=1}^{\infty}$ be a sequence of positive integers where each positive integer occurs exactly once among the n_i (That is N is a

1-1 function from I onto I)

If $\sum_{n=1}^{\infty} a_n$ is a series of real numbers and if $b_i = a_{n_i}$ $(i \in I)$, then $\sum_{i=1}^{\infty} b_i$ is called a rearrangement of $\sum_{n=1}^{\infty} a_n$.

Example :-

Consider $\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \infty$

(ie) $a_n = \frac{(-1)^{n+1}}{n}$

This can be rearranged as

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \cdots \infty = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \left(-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} \cdots \infty\right)$$

Theorem :

Let $\sum_{n=1}^{\infty} a_n$ be a Conditionally series of real numbers Then for any $x \in R$ there is a rearrangement of $\sum_{n=1}^{\infty} a_n$ Which Converges to x.

Lemma :-

If $\sum_{n=1}^{\infty} a_n$ is a series of nonnegative numbers which converges to $A \in R$; and $\sum_{n=1}^{\infty} b_n$ is a rearrangement of $\sum_{n=1}^{\infty} a_n$ then $\sum_{n=1}^{\infty} b_n$ Converges and $\sum_{n=1}^{\infty} b_n = A$.

Proof:-

For each $N \in I$, let $s_N = b_1 + b_2 + \cdots + b_n$

Since $b_i = a_{n_i}$ for some sequence $\{n_i\}_{i=1}^{\infty}$

We have $b_1 = a_{n_1}$; $b_2 = a_{n_2}, \ldots\ldots b_N = a_{n_N}$

Let $M = \text{Max}(n_1, n_2 \ldots\ldots n_N)$

$S_N \leq a_1 + a_2 + \ldots\ldots$

Thus $\sum_{n=1}^{\infty} b_n$ converges to some $B \in \mathbb{R}$

$$\lim_{n \to \infty} S_n = B$$

by ① $B \leq A \Rightarrow \sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\infty} a_n$

Since $\sum_{n=1}^{\infty} a_n$ is also a rearrangement of $\sum_{n=1}^{\infty} b_n$ we can say $A \leq B$

$\therefore A = B$ (ie) $\sum_{n=1}^{\infty} a_n = A \Rightarrow \sum_{n=1}^{\infty} b_n = A$

$\therefore$ Thus proved //

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4. Theorem :-

If $\sum_{n=1}^{\infty} a_n$ converges absolutely to A, then any rearrangement $\sum_{n=1}^{\infty} b_n$ of $\sum_{n=1}^{\infty} a_n$ also converges absolutely to A.

Proof :-

Let $a_n = p_n + q_n$ Where

$p_n = \text{Max}(a_n, 0)$ and $q_n = \text{Min}(a_n, 0)$

then, both $\sum_{n=1}^{\infty} p_n$ converges to P and

$\sum_{n=1}^{\infty} q_n$ converges to Q $\left(\begin{array}{l}\text{where} \\ Q \leq 0\end{array}\right)$

Thus $A = P + Q$ (since $\sum_{n=1}^{\infty} a_n$ converges to A)

For some $\{n_i\}_{n=1}^{\infty}$, we have

$$b_i = a_{n_i} = p_{n_i} + q_{n_i}$$

Thus $\sum_{i=1}^{\infty} p_{n_i}$ is a rearrangement of

$$\sum_{n=1}^{\infty} q_n$$

Thus $\sum_{i=1}^{\infty} p_{n_i}$ converges to P and $\sum_{i=1}^{\infty} q_{n_i}$

Converges to Q.

Thus $b_{n_i} = p_{n_i} + q_{n_i}$ and

$$\sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty} p_{n_i} + \sum_{i=1}^{\infty} q_i = P + Q = A$$

Since $b_i = p_{n_i} + q_{n_i}$ we have

$$|b_i| \leq |p_{n_i}| + |q_{n_i}| = p_{n_i} - q_{n_i}$$

Thus for any $N \in I$

$$|b_1| + |b_2| + \dots + |b_N| \leq \sum_{i=1}^{N} p_{n_i} - \sum_{i=1}^{N} q_{n_i}$$

$$\leq \sum_{i=1}^{\infty} p_{n_i} - \sum_{i=1}^{\infty} q_{n_i}$$

$$= P - Q$$

The partial some of $\sum_{i=1}^{\infty} |b_i|$ are

bounded above.

by $P - Q$ and hence $\sum_{i=1}^{\infty} |b_i| < \infty$

There fore $\sum b_i$ is absolutely convergent.

Theorem :- 4

If the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely to A, and B respectively, then $AB = C$, where $C = \sum_{n=1}^{\infty} C_n$ the series converges absolutely and $C_n = \sum_{k=0}^{n} a_k . b_{n-k}$ $k = 0, 1, 2, \ldots$

Proof :-

For $k = 0, 1, 2, \ldots\ldots$ we have

$$C_k = a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + \ldots\ldots + a_k b_0$$

$$|C_k| = |a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + \ldots\ldots + a_k b_0|$$

$$\leq |a_0 b_k| + |a_1 b_{k-1}| + |a_2 b_{k-2}| + \ldots + |a_k b_0|$$

$$= |a_0||b_k| + |a_1||b_{k-1}| + \ldots\ldots + |a_k||b_0|$$

$$< (|a_0| + |a_1| + |a_2| + \ldots |a_k|)(|b_0| + |b_1| + |b_2| + \ldots |b_k|)$$

Since, $\sum_{n=0}^{\infty} a_n$ & $\sum_{n=0}^{\infty} b_n$ are absolutely Convergent

$$\sum |a_n| < \infty \quad \text{and} \quad \sum |b_n| < \infty$$

$$\sum_{k=0}^{\infty} |C_k| < \infty$$

$\therefore \sum_{k=0}^{\infty} C_k$ is absolutely Convergent.

To prove :-

$\sum_{k=0}^{\infty} |C_k|$ Converges to C & $C = AB$

$$(a_0 + a_1 + a_2 + \ldots\ldots + a_k)(b_0 + b_1 + b_2 + \ldots\ldots + b_k)$$

$$\Rightarrow a_0 b_0 + a_0 b_1 + a_0 b_2 + \ldots\ldots + a_0 b_k + a_1 b_0 + a_1 b_1 + a_1 b_2 + \ldots\ldots$$
$$+ a_1 b_k + a_2 b_0 + a_2 b_1 + \ldots\ldots + a_2 b_k$$
$$+ a_k b_0 + a_k b_1 + \ldots\ldots + a_k b_k$$

Rearranging the terms :-

$\Rightarrow a_0b_0 + (a_0b_1 + a_1b_0 + a_1b_1) + (a_0b_2 + a_1b_2 + a_2b_2 + a_2b_0 + a_2b_1) + \cdots$

Thus $\sum_{k=0}^{\infty} c_k = a_0b_0 + (a_0b_1 + a_1b_0 + a_1b_1) + (a_0b_2 + a_1b_2 + a_2b_2 + a_2b_0 + a_2b_1) + \cdots$ —— ①

If $A_n = a_0 + a_1 + \cdots + a_n$

$B_n = b_0 + b_1 + \cdots + b_n$

Then $a_0b_0 = A_0b_0$

$a_0b_1 + a_1b_0 + a_1b_1 = (a_0 + a_1)(b_0 + b_1) - A_0B_0$

$= A_1B_1 - A_0B_0$

$a_0b_2 + a_1b_2 + a_2b_2 + a_2b_0 + a_2b_1 = (a_0 + a_1 + a_2)(b_0 + b_1 + b_2) - A_1B_1$

$= A_2B_2 - A_1B_1$

Thus in general for $n \geq 1$

Then n^{th} term of ① is $A_nB_n - A_{n-1}B_{n-1}$

Adding all these we have.

$a_0b_0 + (a_0b_1 + a_1b_0 + a_1b_1) + (a_0b_2 + a_1b_2 + a_2b_2 + a_2b_0 + a_2b_1) + \cdots$

$= \cancel{A_0B_0} + A_1B_1 - \cancel{A_0B_0}$

$\Rightarrow \cancel{A_0B_0} + \cancel{A_1B_1} - \cancel{A_0B_0} + \cancel{A_2B_2} - \cancel{A_1B_1} + A_2B_3 - \cancel{A_2B_2} + \cdots + A_nB_n - A_{n-1}B_{n-1}$

$= A_nB_n$

Thus when n approaches infinity

$\lim_{n \to \infty} A_nB_n = AB$

Thus $\sum_{k=0}^{\infty} c_k$ Converges absolutely to AB.

① Classify as to divergent, Conditionally Convergent or absolutely Convergent.

1) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\cdots$

Ans :-

$$\Sigma a_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\cdots$$

$$a_n = \frac{(-1)^{n+1}}{2n-1}$$

$$\Sigma |a_n| = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\cdots$$

$$|a_n| = \frac{1}{2n-1}$$

Thus $\Sigma |a_n|$ diverges.

Therefore Σa_n Conditionally Converges //

2) $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \cdots\cdots$

$$\sum_{n=0}^{\infty} a_n = \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \cdots$$

$$a_n = (-1)^{n+1} \frac{n}{n+1}$$

$$|a_n| = \frac{n}{n+1}$$

$\Sigma |a_n|$ diverges

$\therefore \Sigma a_n$ conditionally Converges.

3) $1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \cdots\cdots$

$$\{S_n\} = \left\{1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, \cdots\cdots\right\}$$

$\therefore \Sigma a_n$ diverges.

$$\Sigma |a_n| = 1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \cdots\cdots$$

$$= 2 + 2\cdot\frac{1}{2} + 2\cdot\frac{1}{3} + 2\cdot\frac{1}{4} + \cdots\cdots$$

$$= 2\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots\right]$$

Since $1 + \frac{1}{2} + \frac{1}{3} + \dots\dots$ diverges

$\therefore \sum |a_n|$ diverges.