Unit - I

Sets And Functions

Defination:

* A set is a collection of well - defined objects (or) elements called elements.

eg:

1) $N = \{1, 2, 3, \ldots\}$ → Natural numbers
2) $N = \{0, 1, 2, 3, \ldots\}$ → Whole numbers.
3) $N = \{\ldots, -2, -1, 0, 1, 2\ldots\}$ → Integers.
4) $Q = \{p/q \,/\, q \neq 0 : p, q \text{ - integers}\}$ → Rational numbers
5) R - Real numbers
6) C - Complex numbers.

finite set and Infinite Set:

* A set with finite number of elements is called finite set otherwise it is called an infinite set

eg: $A = \{1, 2, 3, 4, 5\}$ $B = \{a, b, c, d\}$

Cardinality:

⇒ Let A be a finite set.

⇒ The number of different elements in the set is called the cardinality

eg:- $A = \{1, 2, 3, 4, 5\}$

$n(A) = 5$

i) **Sub - Set :-**

Let A and B are two sets

then, A is sub- set of B ($A \subseteq B$)

if every elements of A is an element of B

example :-

$A = \{1, 2\}$: $B = \{1, 2, 3, 4\}$

$A \subseteq B$

ii) **equal sets :**

~~A = B~~

Two sets A and B are equal iff and only $A \subseteq B$ and $B \subseteq A$

example:-

$A = \{a, b, c\}$: $B = \{b, c, a\}$

$A \subseteq B$ and $B \subseteq A$

$\Rightarrow A = B$

iii) **null - sets :- (or) empty set :-**

A set having no elements is called a null set.

example :-

$A = \{x \in N \mid 1 < x < 2\}$

$= \{\}$

iv) **Single - ton set :-**

A set having exaclty one element is called a singleton set.

example :-

$A = \{a\}$

$B = \{2\}$

v) **Universal set :**

A set is called a Universal set if it is the superset of All the sets.

examgle :

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{1, 3, 5, 7, 9\}$

$B = \{2, 4, 6, 8\}$

$C = \{1, 4, 7, 9, 10\}$

$D = \{2, 4, 5, 7, 8, 9\}$

Operation on sets :

* Union
* intersection
* symertric
* diffrence
* compliments

i) Union :-

Let A and B are two sets.

$$A \cup B = \left\{ x \;\middle|\; \begin{matrix} x \in A \\ (\text{or}) \quad x \in B \end{matrix} \right\}$$

example :-

$A = \{1, 2, 3\}$: $B = \{4, 5, 6\}$

$A \cup B = \{1, 2, 3, 4, 5, 6\}$

ii) Intersection :-

Let A and B are two sets

$$A \cap B = \left\{ x \;\middle|\; \begin{matrix} x \in A \\ \text{and} \quad x \in B \end{matrix} \right\}$$

example :-

i) $A = \{1, 2, 3\}$: $B = \{4, 5, 6\}$

$A \cap B = \{\}$

ii) $A = \{1, 2, 3, 4, 5\}$: $B = \{2, 3, 6\}$

$A \cap B = \{2, 3\}$

~~iii) symmetric :~~

iii) Complements :-

Let U be the universal set and A be any set in U.

$$A^c = \left\{ x \;\middle|\; \begin{matrix} x \in U \\ x \notin A \end{matrix} \right\}$$

$A \cup A^c = U$

$A \cap A^c = \phi$

example :-

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{1, 3, 5, 7, 9\}$: $A^c = \{2, 4, 6, 8, 10\}$

iv) **difference** :-

Let A and B are two sets.

$$A - B = \{ x \mid x \in A \text{ and } x \notin B \}$$

$$B - A = \{ x \mid x \in B \text{ and } x \notin A \}$$

example :

$A = \{a, b, c, d, e, f, g\}$: $B = \{a, b, c, d, h, i, j\}$

$A - B = \{e, f, g\}$: $B - A = \{h, i, j\}$

v) **Symetric** :- (power set) :-

set of all subsets.

Let A, B any set, the set of sub a is called a power set. And it is donoted by P(A).

if the set A has m elements then the power set P(A) has 2^m elements.

example :

$A = \{a, b, c\}$

$n(A) = 2^3 = 8$

$P(A) = \{\{a, b, c\}, \phi, \{a\} \{b\} \{c\} \{a, b\} \{b, c\} \{c, a\}$

Basic Law's of set theory :

1) Associative Law :

A, B, C

$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

2) Commulative Law :-

$A \cup B = B \cup A$

$A \cap B = B \cap A$

3) Distributive Law :

A, B, C

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4) Demorgan's Law :

$(A \cup B)^c = A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$

5) Idempotent Law :-

$A \cup A = A$

$A \cap A = A$

6) negation Law:

$A \cup A^c = U$

$A \cap A^c = \phi$

7) Empty set

$A \cup \phi = A$

$A \cap \phi = \phi$

8) $A \cup U = U$

$A \cap U = A$

9) $(A^c)^c = A$

10) $A \subseteq B$

$A \cap B = A$

$A \cup B = B$

Problem : 1

1 Describe the following sets of real numbers geomitrically !

1. $A = \{x / x < 7\}$

Solution :-

The set A describes the ray extending from $-\infty$ to 7 but not induding 7.

2. $B = \{x / |x| \geq 2\}$

solution :-

$x \geq 2 \Rightarrow 2 \leq x \leq \infty$

$-x \geq 2 \Rightarrow x \leq -2$

$\Rightarrow -\infty \leq x \leq -2$

represents a colsed ray from 2 to ∞ and a colesed ray from $-\infty$ to -2

2) $C = \{x / |x| = 1\}$

solution :

$|x| = 1 \Rightarrow x = \pm 1$

$x = 1, -1$ are the points

~~Problem~~ Problem - 1 :

1) Let P be the set of prime integers.
Which of the following are true ?

i) $7 \in P$

ii) $9 \in P$

iii) $11 \notin P$

iv) $7, 547, 193.63, 317 \in P$

Ans :

i) $7 \in P$

$7 \in P$ is true

Since 7 is a prime integer

ii) $9 \in P$ is false

Since 9 is a not a prime interger and its divisible by 3.

iii) $11 \notin P$

$11 \notin P$ is false

Since 11 is a prime interger it belongs to the set P.

iv) $7, 547, 193.63, 317 \in P$

$193.63 \notin P$

Since it not a prime Interger.

$\therefore 7, 547, 193.63, 317 \in P$ is false

Problem - 2 :

2) Let $A = \{1, 2, \{3\}, \{4, 5\}\}$

are the following T (or) F ?

i) $1 \in A$ 2) $3 \in A$ 3) How many elemants in A ?

Answer :

1) $1 \in A$

$1 \in A$ is true

2) $3 \in A$

$3 \in A$ is false

but $\{3\} \in A$

3) $n(A) = 4$.

Problem-(3):

Let A be the set of letter's in the word "trivial".

$$A = \{a, i, l, r, t, v\}.$$

Let B be the set of letter's in the word "difficult". Find $A \cup B$, $A \cap B$, $A - B$, $B - A$.

Solution:

$$A = \{a, i, l, r, t, v\}$$

$$B = \{d, i, f, c, u, l, t\}$$

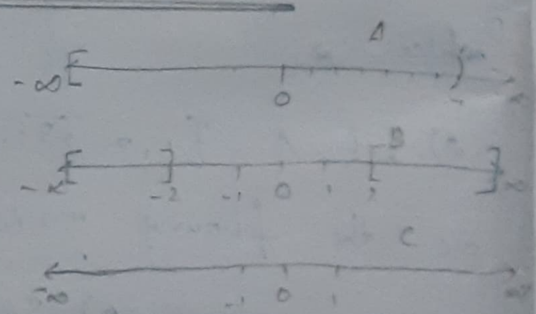
$$A \cup B = \{a, i, l, r, t, v, d, f, c, u\}$$

$$A \cap B = \{i, l, t\}$$

$$A - B = \{a, r, v\}$$

$$B - A = \{d, f, c, u\}$$

Problem (4):



for the set $A = \{x \mid x < 7\}$

$$B = \{x \mid |x| \geq 2\}$$

$$C = \{x \mid |x| = 1\}$$

describe; $A \cap B$, $B \cap C$, $A \cap C$

A — $[-\infty, 7)$

B — $[-\infty, -2] \cup [2, \infty)$

C — ± 1

Ans:

$$A \cap B = [-\infty, -2] \cup [2, 7)$$

$$B \cap C = \phi$$

$$A \cap C = \pm 1$$

Problem : (5)

For any set's A , B , C Prove that :

$A \cup (B \cup C) = (A \cup B) \cup C$ → Associative Law.

Proof :

$A \cup (B \cup C) \subseteq (A \cup B) \cup C$ and

$(A \cup B) \cup C \subseteq A \cup (B \cup C)$

Let $x \in A \cup (B \cup C)$

$\Leftrightarrow x \in A$ (or) $x \in B \cup C$

$\Leftrightarrow x \in A$ (or) $x \in B$ (or) $x \in C$

$\Leftrightarrow [x \in A$ (or) $x \in B]$ (or) $x \in C$

$\Leftrightarrow x \in A \cup B$ (or) $x \in C$

$\Leftrightarrow x \in (A \cup B) \cup C$

$A \cup (B \cup C) \subseteq (A \cup B) \subseteq C$

and $(A \cup B) \cup C \subseteq A \cup (B \cup C)$

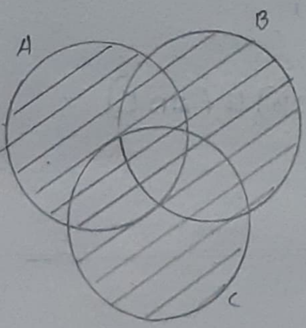
$\therefore A \cup (B \cup C)$

$= (A \cup B) \cup C$

$(A \cup B) \cup C = A \cup (B \cup C)$

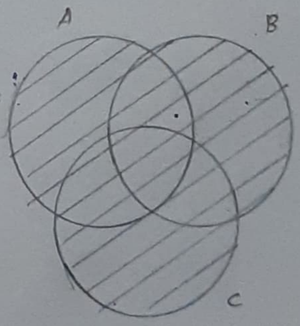
Proof :

LHS :



$(A \cup B) \cup C$

RHS :



$A \cup (B \cup C)$

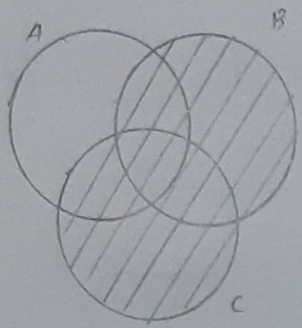
$(A \cup B) \cup C = A \cup (B \cup C)$

Problem (6):

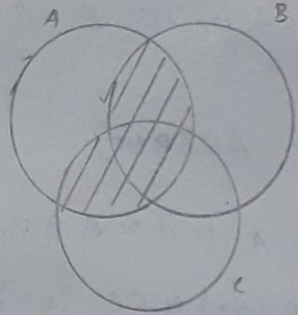
Proof that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof:

L.H.S:

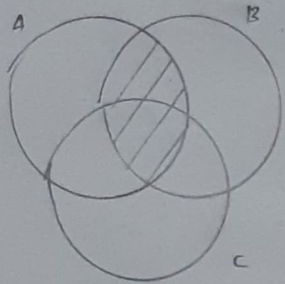


$B \cup C$

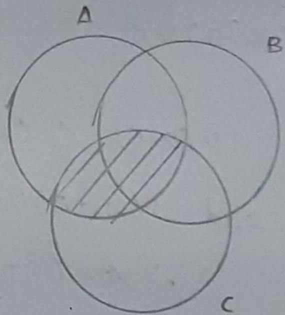


$A \cap (B \cup C)$

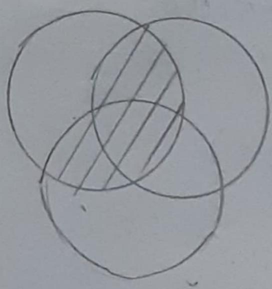
RHS:



$A \cap B$



$A \cap C$



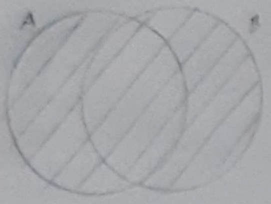
$(A \cap B) \cup (A \cap C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

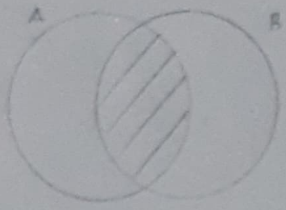
Proof that $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$

Proof

LHS :



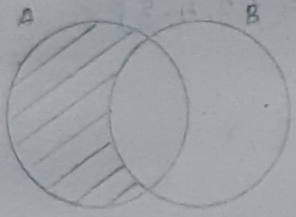
$A \cup B$



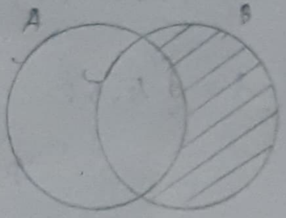
$A \cap B$

$(A \cup B) - (A \cap B)$

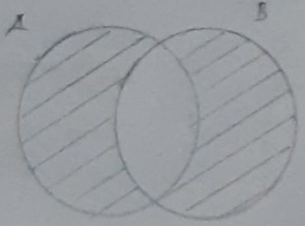
RHS :



$A - B$



$B - A$



$(A - B) \cup (B - A)$

$$(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$$

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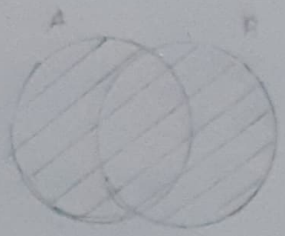
1) $(A \cup B) - A = B$

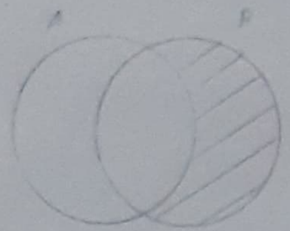
Proof :

L.H.S

R.H.S



$A \cup B$



$(A \cup B) - A$

B

$$(A \cup B) - A \neq B$$

$$L.H.S \neq R.H.S$$

example's :

$A = \{1, 2, 3\}$, $B = \{2, 4, 5\}$

L.H.S :

$A \cup B = \{1, 2, 3, 4, 5\}$

$(A \cup B) - A = \{4, 5\}$

R.H.S :

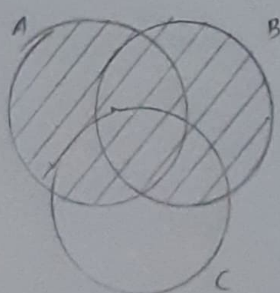
$B = \{2, 4, 5\}$

$$A \cup B - A \neq B$$

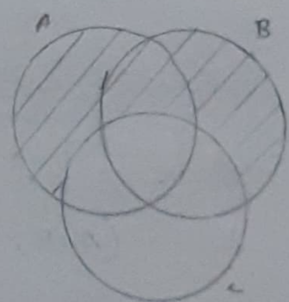
True (or) false :

2) $(A \cup B) - C = A \cup (B - C)$

L.HS :-

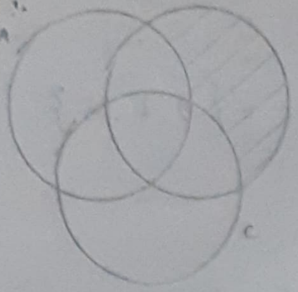


$A \cup B$

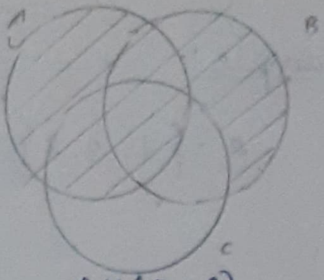


$(A \cup B) - C$

R.H.S :-



B - C



$A \cup (B - C)$

$\therefore (A \cup B) - C = A \cup (B - C)$

example :-

2) $(A \cup B) - C = A \cup (B - C)$

$A = \{1, 2, 3\}$ $B = \{2, 3, 4\}$ $C = \{2, 4, 5\}$

LHS :-

$A \cup B = \{1, 2, 3, 4\}$

$(A \cup B) - C = \{1, 3\}$

RHS :-

$B - C = \{3\}$

$A \cup (B - C) = \{1, 2, 3\}$

$\therefore (A \cup B) - C \neq A \cup (B - C)$

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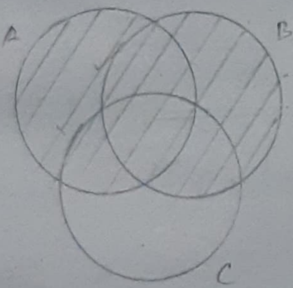
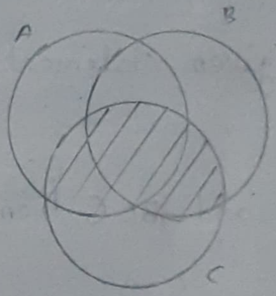
3) $(A \cup B) \cap C = A \cup (B \cap C)$

RHS :

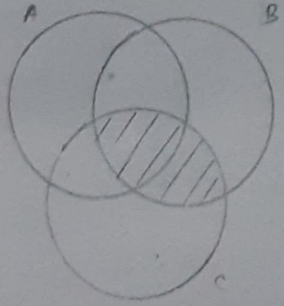


$A \cup B$

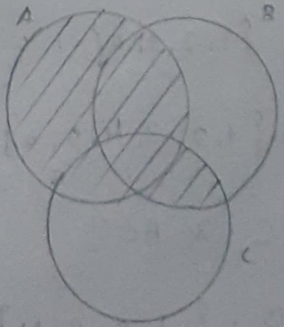


$(A \cup B) \cap C$

RHS.



$B \cap C$



$A \cup (B \cap C)$

$(A \cup B) \cap C \neq A \cup (B \cap C)$

example :-

$A = \{1, 2, 3\}$ $B = \{2, 3, 4\}$, $C = \{2, 4, 5\}$

LHS :

$A \cup B = \{1, 2, 3, 4\}$

$(A \cup B) \cap C = \{2, 4\}$

RHS :-

$B \cap C = \{2, 4\}$

$A \cup (B \cap C) = \{1, 2, 3, 4\}$

$$(A \cup B) \cap C \neq A \cup (B \cap C)$$

True (or) flase :-

a) $A \subset B$ and $B \subset C$, then $A \subset C$

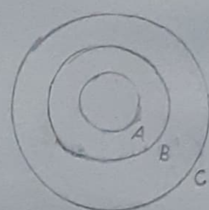
Ex :-

Let $A = \{1, 2\}$

$B = \{1, 2, 3, 4\}$

$C = \{1, 2, 3, 4, 5, 6\}$

Soln :-



The given statement is true.

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b) $A \subset C$ and $B \subset C$, then $A \cup B \subset C$

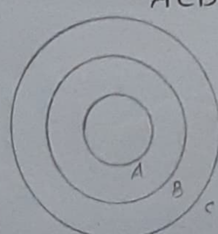
Ex :-

Let $A = \{1, 2\}$

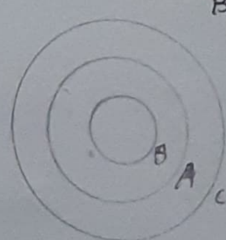
$B = \{1, 2, 3, 4\}$

$C = \{1, 2, 3, 4, 5, 6\}$

Case (i)
$A \subset B$



Case (ii)
$B \subset A$



$A \subset C$ & $B \subset C$

$A \cup B = \{1, 2, 3, 4\} \subset C$

$\therefore$ The given statement is true.

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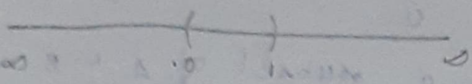
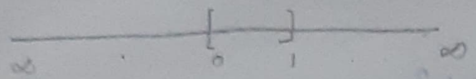
c) $[0, 1] \supset (0, 1)$

$(0, 1)$



$[0, 1]$



$(0, 1) \subset [0, 1]$

In $(0, 1)$ the set contains the value's between 0 and 1

In $[0, 1]$ the set contains the value's between 0 and 1 also including the value 0 and 1

$\therefore (0, 1) \subset [0, 1]$.

$\therefore$ The given statement true.

Functions:

Definition: [Cartesian Product]

If A and B are two sets, then.

$$A \times B = \left\{ (a, b) \,\middle|\, \begin{matrix} a \in A \\ b \in B \end{matrix} \right\}$$

example:

$A = \{a, b\}$, $B = \{1, 2, 3\}$

$A \times B = \{ (a, 1) (a, 2) (a, 3) (b, 1) (b, 2) (b, 3) \}$

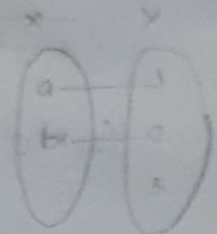
$B \times A = \{ (1, a) (2, a) (3, a) (1, b) (2, b) (3, b) \}$

$A \times B \neq B \times A$

Difinition : Function :

Let A and B be any two sets A function f from A into B is a subset of $A \times B$ with the property that each $a \in A$ belongs to precisely one pair (a, b)

Domain $= X = \{a, b\}$



Co-Domain $= Y = \{1, 2, 3\}$

Range $= \{1, 2\}$

Range $=$ Co-Domain.

Instead of $(x, y) \in f$ we usully write $y = f(x)$

Then y is called the image of x under f.

The set A is called the domain of f.

The range of f is the set $\{ b \in B \mid b = f(a) \text{ for some } a \}$

such a function is sometimes called a mapping of A into B.

---

example :

$$f = \{ (x, x^2) \mid -\infty < x < \infty \}$$

$$f(x) = x^2$$

Domain $= (-\infty, \infty)$

Range ~~Co-Domain~~ $= [0, \infty)$

$f(2) = 2^2 = 4$

$f(\{x / x^2 = 9\}) = 9$

$f([0, 3)) = [0, 9)$

$f^{-1}(4) = \{-2, 2\}$

$f^{-1}(-7) = \phi$

Hint :

An equation $f(x) = 1 + x^3$

does not define a function

Until the domain is explicitly specified.

example :

$f(x) = 1 + x^3 \quad , \quad 1 \le x \le 3$

$g(x) = 1 + x^3 \qquad 1 \le x \le 4$

are different functions.

Definition : Restriction :

Suppose f and g are two functions with return to domains X and Y ;

If $X \subset Y$ and if $f(x) = g(x) \ \forall x \in X$

Then g is said to be an extension of f to y

(or) f is the restriction of g to x.

example :

$f(x) = x \quad , \quad 0 \le x < \infty$

$g(x) = |x| \quad , \quad -\infty < x < \infty$

f is a restriction of g

Hence g is an extension of f.

example :

$f(x) = \sin x \quad ; \quad 0 \le x \le 2\pi$

$g(x) = \sqrt{1 - \cos^2 x} \quad , \quad -\infty < x < \infty$

f is a restriction of g

Hence g is an extension of f.

Theorem 1 : *

If $f : A \to B$ and if $X \subset B$, $Y \subset B$

then $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$

Proof :

P.T $f^{-1}(X \cup Y) \subseteq f^{-1}(X) \cup f^{-1}(Y)$

Let $a \in f^{-1}(X \cup Y)$

$\Rightarrow f(a) \in X \cup Y$

$\Rightarrow f(a) \in X$ (or) $f(a) \in Y$

$\Rightarrow a \in f^{-1}(X)$ (or) $a \in f^{-1}(Y)$

$\Rightarrow a \in f^{-1}(X) \cup f^{-1}(Y)$

$\therefore f^{-1}(X \cup Y) \leq f^{-1}(X) \cup f^{-1}(Y)$ —— ①

Let $b \in f^{-1}(X) \cup f^{-1}(Y)$

$\Rightarrow b \in f^{-1}(X) \cup f^{-1}(Y)$

$\Rightarrow f(b) \in X$ (or) $f(b) \in Y$

$\Rightarrow f(b) \in X \cup Y$

$\Rightarrow b \in f^{-1}(X \cup Y)$

$f^{-1}(X) \cup f^{-2}(Y) \leq f^{-1}(X \cup Y)$ —— ②

From ① & ②

$f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$

Thearem 2 :

If $f : A \to B$ and if $X \subset B$, $Y \subset B$, then

$f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$

Proof :

P.T $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$

Let $x \in f^{-1}(X \cap Y)$

$\Rightarrow f(x) \in X \cap Y$

$\Rightarrow f(x) \in X$ and $f(x) \in Y$

$\Rightarrow x \in f^{-1}(X)$ and $x \in f^{-1}(Y)$

$\Rightarrow x \in f^{-1}(X) \cap f^{-1}(Y)$

$\therefore f^{-1}(X \cap Y) \subseteq f^{-1}(X) \cap f^{-1}(Y)$ — ①

Let $y \in f^{-1}(X) \cap f^{-1}(Y)$

$\Rightarrow y \in f^{-1}(X)$ and $y \in f^{-1}(Y)$

$\Rightarrow f(y) \in X$ and $f(y) \in Y$

$\Rightarrow f(y) \in X \cap Y$

$\Rightarrow y \in f^{-1}(X \cap Y)$

$\therefore f^{-1}(X) \cap f^{-1}(Y) \subseteq f^{-1}(X \cap Y)$ — ②

From ① & ②

$f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$.

Problem's :

(1) If $A = [0, 1]$ and $B = [1, 2]$

Verify theorem (1) & (2)

Ans :

Verify $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

$A = [0, 1]$

$f^{-1}(A) = [-1, 0]$

$B = [1, 2]$

$f^{-1}(B) = [-2, -1]$

$A \cup B = [0, 2]$

$f^{-1}(A \cup B) = [-2, 0]$ —— ①

$A \cap B = 1$

$f^{-1}(A \cap B) = -1$ —— ②

$f^{-1}(A) \cup f^{-1}(B) = [-2, 0]$ —— ③

$f^{-1}(A) \cap f^{-1}(B) = -1$ —— ④

From ①, ②, ③, ④

$f^{-1}(A \cup B) = [-2, 0] = f^{-1}(A) \cup f^{-1}(B)$

$f^{-1}(A \cap B) = -1 = f^{-1}(A) \cap f^{-1}(B)$

Thearem 3:

If $f : A \to B$ and $X \subset A$, $Y \subset A$

then $f(X \cup Y) = f(X) \cup f(Y)$

Proof:

P.T. If $a \in f(X \cup Y)$, then $a = f(b)$ for some $b \in X \cup Y$

$\Rightarrow b \in X$ (or) $b \in Y$

Thus either $a \in f(X)$ or $a \in f(Y)$

$\Rightarrow a \in f(X) \cup f(Y)$

$f(X \cup Y) \subseteq f(X) \cup f(Y)$ — ①

Let $c \in f(X) \cup f(Y)$

$\Rightarrow c \in f(X)$ (or) $c \in f(Y)$

$\Rightarrow$ c is the image of some point in X.

(or)

c is the image of some point in Y.

$\Rightarrow$ c is the image of some point [in $X \cup Y$]

$\Rightarrow c \in f(X \cup Y)$

$\therefore f(X) \cup f(Y) \subseteq f(X \cup Y)$ — ②

From ① & ②

$f(X \cup Y) = f(X) \cup f(Y)$

Note:

$f(X \cap Y) \neq f(X) \cap f(Y)$

problems, Theorem 4 :-

Consider the sine function defined by.

$f(x) = \sin x \quad , \quad -\infty < x < \infty$

Let $A = [0, \pi/6] \quad , B = [\pi/6, \pi]$

Verify (i) $f(A \cup B) \neq f(A) \cup f(B)$

(ii) $f(A \cap B) \neq f(A) \cap f(B)$

Answer:-

i) Domain $= [-\infty, \infty] = R$.

Range $= [-1, 1]$

$f(A) = f([0, \pi/6]) = [0, 1/2]$

$f(B) = f([\pi/6, \pi/2]) = [1/2, 1]$

$A \cup B = [0, \pi/2]$

$f(A \cup B) = [0, 1]$

$\therefore f(A) \cup f(B) = [0, 1] = f(A \cup B)$

~~Consider the sin function defined by~~

~~$f(x) = \sin x \quad , -\infty < x < \infty$~~

Theorem : 4

Example :-

(i) Let $f: \{1, 2\} \to \{1\}$

be given by $f(1) = 1$, $f(2) = 1$

and Let $A = \{1\}$, $B = \{2\}$

Verify (i) $f(A \cup B) = f(A) \cup f(B)$

(ii) $f(A \cap B) = f(A) \cap f(B)$

Answer:-

$f(A) = \{1\}$

$f(B) = \{1\}$

$f(A \cup B) = \{1\}$

$f(A \cap B) = \{\phi\}$

$f(A) \cup f(B) = \{1\} = f(A \cup B)$

$f(A) \cap f(B) = \{1\} \neq f(A \cap B)$

Definition :

The composition of function's :

If $f : A \to B$ and $g : B \to C$

we define the function gof by $(gof)(x) = g(f(x))\ \forall x \in$

example :

$f(x) = 1 + \sin x \quad , \quad -\infty < x < \infty$

$g(x) = x^2 \quad , \quad 0 < x < \infty$

$$(gof)(x) = g(f(x))$$
$$= g(1 + \sin x)$$
$$= (1 + \sin x)^2$$
$$= 1 + \sin^2 x + 2 \sin x \,, \quad -\infty < x < \infty$$

Real Valued function :

Definition :

(1) If $f : A \to R$ and $g : A \to R$

$f + g$ is difined as $(f + g)(x) = f(x) + g(x)$. $\forall x \in A$

(2) If $f : A \to R$ and $g : A \to R$

then $f - g$ is difined as $(f - g)(x) = f(x) - g(x)$ $\forall x \in A$

(3) If $f : A \to R$ and $g : A \to R$,

then fg is defined as $(fg)(x) = f(x)g(x)$

(4) If $f : A \to R$ and $g : A \to R \ni g(x) \neq 0 \quad \forall x \in A$

then $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \forall x \in A$.

5) If $f : A \to R$ and c is any real number

then $(cf)(x) = c(f(x)) \ \forall x \in A$

6) If $f : A \to R$ and $g : A \to R$,

then, $\text{Max}(f, g)(x) = \text{Max}[f(x), g(x)] \ \forall x \in A$

$\text{Min}(f, g)(x) = \text{Min}[f(x), g(x)]$.

Definition : Modules of functions :

If $f : A \to R$. then $|f|$ is the function defined by

$|f|(x) = |f(x)| \quad \forall x \in A$.

If a, b are real numbers,

$$\max(a, b) = \frac{|a-b| + a + b}{2}$$

$$\min(a, b) = \frac{-|a-b| + a + b}{2}$$

For real valued function f and g,

$$\max(f, g) = \frac{|f-g| + f + g}{2}$$

$$\min(f, g) = \frac{-|f-g| + f + g}{2}$$

Characteristic function :

If $A \subset S$, then ψ_A is defined as

$$\psi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

Theorem : 5 : *

If A and B are any 2 subsets of S.

then,

(1) $\psi_{A \cup B} = \text{Max}(\psi_A, \psi_B)$

(2) $\psi_{A \cap B} = \text{Min}(\psi_A, \psi_B)$

(3) $\psi_{A - B} = \psi_A - \psi_B$

(4) $\Psi_{A^c} = 1 - \Psi_A$ (6) $\Psi_\phi = 0$

(5) $\Psi_S = 1$

1. Proof: $\Psi_{A\cup B} = \text{Max}\,(\Psi_A \cdot \Psi_B)$

(1) Let $x \in A \cup B$

$\Rightarrow \Psi_{A\cup B}(x) = 1$

$x \in A \cup B \Rightarrow x \in A$ (or) $x \in B$

$\Rightarrow \Psi_A(x) = 1$ (or) $\Psi_B(x) = 1$

$\Rightarrow \text{Max}\,[\Psi_A(x), \Psi_B(x)]$

$\Rightarrow 1$

$\therefore \Psi_{A\cup B} \Rightarrow \text{Max}\,[\Psi_A\ \Psi_B]$

$\Rightarrow 1$

(2) Let $x \notin A \cup B$

$\Psi_{A\cup B}(x) = 0$

$x \notin A\cup B \Rightarrow x \notin A$ (or) $x \notin B$

$\Rightarrow \Psi_A(x) = 0$ (or) $\Psi_B(x) = 0$

$\Rightarrow \text{Max}\,[\Psi_A(x)\ \Psi_B(x)] = 0$

$\therefore \Psi_{A\cup B} = \text{Max}\,[\Psi_A, \Psi_B].$

$= 0$

2. Proof: $\Psi_{A\cap B} = \text{Min}\,(\Psi_A, \Psi_B)$

Let $x \in A \cap B$

$\Rightarrow \Psi_{A\cap B}(x) = 1$

$x \in A \cap B \Rightarrow x \in A$ and $x \in B$

$\Rightarrow \Psi_A(x) = 1$ and $\Psi_B(x) = 1$

$\Rightarrow \text{Min}\,[\Psi_A(x), \Psi_B(x)] = 1$

$\Psi_{A\cap B} = \text{Min}\,[\Psi_A, \Psi_B]$

Let $x \in A \cap B$

$\Rightarrow \psi_{A \cap B}(x) = 0$

$x \notin A \cap B \Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow \psi_A(x) = 0$ and $\psi_B(x) = 0$

$\Rightarrow \text{Min}[\psi_A(x), \psi_B(x)] = 0$

$\psi_{A \cap B} = \text{Min}[\psi_A, \psi_B]$

---

3. Proof: $\psi_{A-B} = \psi_A - \psi_B$

Let $x \in A - B$

$\Rightarrow \psi_{A-B}(x) = 1$

$x \in A - B \Rightarrow x \in A$ but $x \notin B$

$\Rightarrow \psi_A(x) = 1$ but $\psi_B(x) = 0$

$\psi_A(x) - \psi_B(x) = 1 - 0 = 1$

$\psi_{A-B} = \psi_A - \psi_B$

Let $x \notin A - B$

$\Rightarrow \psi_{A-B}(x) = 0$

$x \in A - B \Rightarrow x \in$ both A and B (or)

$x \notin$ both A and B

~~3. Proof~~

$\Rightarrow \psi_A(x) = 1$

$\psi_{(B)}(x) = 1$

(or)

$\psi_A(x) = 0$,

$\psi_{A'}(x) = 1$

$\Rightarrow \psi_A(x) - \psi_B(x) = 0$

$\psi_{A-B} = \psi_A - \psi_B$

4. Proof: $\Psi_{A^c} = 1 - \Psi_A$

Let $x \in A^c \Rightarrow \Psi_{A^c}(x) = 1$

$x \in A^c = x \in A$

$\Rightarrow \Psi_A(x) = 0$

$\Rightarrow 1 - \Psi_A(x) = 1 - 0 = 1$

$\Psi_{A^c} = 1 - \Psi_A$

Let $x \notin A^c \Rightarrow \Psi_{A^c}(x) = 0$

$x \notin A^c \Rightarrow x \in A$

$\Rightarrow \Psi_A(x) = 1$

$\Rightarrow 1 - \Psi_A(x) = 0$

$\Psi_{A^c} = 1 - \Psi_A$

---

5. Proof: $\Psi_S = 1$

$\Psi_S = 1$

since S is the Universal set,

$\Psi_S = 1$

---

6. proof: $\Psi_\phi = 0$

We know that, $\Psi_{A^c} = 1 - \Psi_A$

Take $A = S$

$\Psi_{S^c} = 1 - \Psi_S$

$\Psi_\phi = 1 - \Psi_S$

$= 1 - 1$

$= 0$

Example's :- (2)

1) Let $f(x) = 2x \ (-\infty < x < \infty)$,

Can you think of function's g and h which satistify the equations.

$$g \circ f = 2gh$$
$$h \circ f = h^2 - g^2$$

Answer:

i) $g \circ f = 2gh$

Let, $g(x) = \sin x, \quad -\infty < x < \infty$

$h(x) = \cos x, \quad -\infty < x < \infty$

$$(g \circ f)(x) = g(f(x))$$
$$= g(2x)$$
$$= \sin 2x$$
$$= 2 \sin x \cos x$$
$$= 2\, g(x)\, h(x)$$
$$g \circ f = 2gh$$

ii) $h \circ f = h^2 - g^2$

$\sin 2\theta = 2 \sin\theta \cos\theta$
$\cos 2\theta = \cos^2\theta - \sin^2\theta$

Let. $(h \circ f)(x) = h(f(x))$

$$= h(2x)$$
$$= \cos 2x$$
$$= \cos^2 x - \sin^2 x$$
$$= h^2(x) - g^2(x)$$
$$h \circ f = h^2 - g^2$$

Example's :- (2)

If $f(x) = x^2$ $(-\infty < x < \infty)$ and ψ, is the characteristic function of $[0,9]$ of what subset of R is ψ of R is $\psi o f$ the characteristic function?

Ans :-

$$f(x) = x^2 \quad , \quad -\infty < x < \infty$$

$$\psi(x) = \begin{cases} 1 & , \ x \in [0,9] \\ 0 & ; \ x \notin [0,9] \end{cases}$$

$$(\psi o f)(x) = \psi(f(x))$$

$$= \psi(x^2)$$

$$(\psi o f)(x) = \begin{cases} 1 & , \ x \in [-3,3] \\ 0 & , \ x \notin [-3,3] \end{cases}$$

Equivalence and Countability

Definition :- [One to One function]

If $f : A \to B$ then f is called 1-1 $\forall$

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \ \forall \ x_1, x_2 \in A$

Example :-

(i) $f(x) = x^2$, $(-\infty, \infty)$

$f(2) = 4$

$f(-2) = 4$

f is not 1-1

(ii) $f(x) = x^2$, $(0, \infty)$

is a 1-1 function.

[for every x, there exist a unique $f(x)$]

(iii) $f(x) = e^x, \quad (-\infty, \infty)$

$f(x_1) = f(x_2)$

$\Rightarrow e^{x_1} = e^{x_2}$

$\Rightarrow x_1 = x_2$

e^x is a 1-1 function.

(iv) $f(x) = e^{x^2}, \quad (-\infty, \infty)$

$f(x_1) = f(x_2)$

$\Rightarrow e^{x_1^2} = e^{x_2^2}$

$\Rightarrow x_1^2 = x_2^2$

$\Rightarrow x_1 = \pm x_2$

$f(x) = e^{x^2}$ is not 1-1

(v) $f(x) = \cos x, \quad (0, \pi)$

$f(x_1) = f(x_2)$

$\Rightarrow \cos x_1 = \cos x_2$

$\Rightarrow x_1 = x_1 \quad \forall x_1, x_2 \in (0, \pi)$

$\therefore f(x) = \cos x$ is 1-1

(vi) $f(x) = \sin x, \quad (0, \pi)$

$\sin 0 = 0$

$\sin \pi = 0$

$\therefore f(x) = \sin x$ is not 1-1 in $[0, \pi)$

(7) $f(x) = ax + b$, $-\infty < x < \infty$, $a, b \in \mathbb{R}$

$f(x_1) = f(x_2)$

$\Rightarrow ax_1 + b = ax_2 + b$

$ax_1 - ax_2 = 0$

$\Rightarrow a(x_1 - x_2) = 0$

$\Rightarrow a = 0$ (or) $x_1 - x_2 = 0$

If $a = 0$, $f(x) = b$ is a constant function but constant function is not 1-1

If $a \neq 0$, $f(x) = ax + b$ is 1-1.

(8) If $f: A \to B$ and $g: B \to C$ both f and g are 1-1, then gof is also 1-1

Proof:

$(gof)(x_1) = (gof)(x_2)$

$\Rightarrow g(f(x_1)) = g(f(x_2))$

$\Rightarrow f(x_1) = f(x_2)$ ($\because g$ is 1-1)

$\Rightarrow x_1 = x_2$

$\therefore gof$ is 1-1

Inverse Function:

Definition:

If $f: A \to B$ and f is 1-1, then the function f^{-1} is called the inverse function for f is defined as follows.

If $f(a) = b$, then $f^{-1}(b) = a$. b is the range of f

Thus the domain of f^{-1} is the range of f and the range of f^{-1} is the domain of f.

Example :

(1) If $g(x) = x^2$, $(0, \infty)$

then $g^{-1}(x) = \sqrt{x}$

(2) $h(x) = e^x$ $(-\infty, \infty)$

then $h^{-1}(x) = \log x$.

1-1 Correspondence :

Definition :

If $f: A \to B$ and f is 1-1 then f is called a 1-1 correspondence between A and B.

If there exist a 1-1 correspondence between the sets A and B, then A and B are equivalent sets.

NOTE :

(1) Any 2 sets A and B are equivalant if $n(A) = n(B)$

(2) Every set A is equivalent to itself.

(3) If A and B equivalent then B and A are equivalent.

(4) A, B, C

$n(A) = n(B)$: $n(B) = n(C)$

$n(A) = n(C)$

If A and B are equivalent and B and C are equivalent, then A and C are equivalent.

Infinite Set:

The set A, is said to be infinite if for each positive integer n,

A contains a subset with precisely n elements.

(ex) (1) $I = \{1, 2, 3, \ldots\}$

(2) The set R of real number is an infinite set.

Finite set:

A set that is not infinite is called finite sets.

Countable [or] denumberable:

Definition:

⇒ The set A is said to be countable if A is equivalent to the set of positive integers (I).

⇒ An uncountable set which is not countable.

⇒ Thus A is countable if there exist a 1-1 function f from I onto A.

⇒ The elements of A are then the images $f(1), f(2), \ldots$ of the positive integers (I)

$$A = \{f(1), f(2), f(3), \ldots\}.$$

NOTE :-

Hence saying that A is countable means that its elements can be cantined.

EXAMPLE 1.

(2) If A and B are countable, then $A \cup B$ is also countable.

Proof :- Let,

$A = \{a_1, a_2, a_3, \dots\}$

$B = \{b_1, b_2, b_3, \dots\}$

then, $A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, \dots\}$.

Obtained by remaving any b which occurs among the a is so that the same element in $A \cup B$ is not counted twice.

$\therefore A \cup B$ is countable.

Theorem 6 :-

If $A_1, A_2, \dots$ are countable sets, then $\bigcup_{n=1}^{\infty} A_n$ is also countable.

Proof :-

Let $A_1 = \{a_1^1, a_2^1, a_3^1 \dots\}$

$A_2 = \{a_1^2, a_2^2, a_3^2 \dots\}$

$A_3 = \{a_1^3, a_2^3, a_3^3 \dots\}$

$A_n = \{a_1^n, a_2^n, a_3^n \dots\}$

$\vdots$

So that a_n^j is the k^{th} element of the set A_j. Define the height of a_n^j by $j+k$

Thus a_1^1 is the only element of height 2. a_2^1 and a_1^2 are the Only element's of height 3, and so on.

Since for any positive integer $m \geq 2$ there are only $m-1$ elements of height m.

We may arrange (count) the elements of $\bigcup_{n=1}^{\infty} A_n$ according to their height as,

$$a_1^1,\ a_1^2,\ a_2^1,\ a_3^1,\ a_2^2,\ a_1^3, \dots\dots$$

Pictorially we are listing the elements of $\bigcup_{n=1}^{\infty} A_n$ in the following array

$$\begin{array}{llll}
a_1^1, & a_2^1, & a_3^1, & a_4^1 \ \dots\dots \\
a_1^2, & a_2^2, & a_3^2, & a_4^2, \ \dots\dots \\
a_1^3, & a_2^3, & a_3^3, & a_4^3, \ \dots\dots \\
a_1^4, & a_2^4, & a_3^4, & a_4^4, \ \dots\dots
\end{array}$$

The fact that this counting scheme, eventually occurs every a_n^j this prove that $\bigcup_{n=1}^{\infty} a_n$ is countable.

Theorem 7:-

Prove that the set of rational numbers is countable.

Proof: Let, $E_n = \left\{ \frac{0}{n}, \frac{1}{n}, \frac{-1}{n}, \frac{2}{n}, \frac{-2}{n} \dots\dots \right\}$ be the set of all rational number's.

Then the set of rations = $\bigcup_{n=1}^{\infty} E_n$

Each E_n is clearly equivalent to the set of integers.

Hence each E_n is countable

$\therefore$ From Theorem 6 $\bigcup_{n=1}^{\infty} E_n$ is countable.

$\therefore$ set of rational numbers is countable.

Theorem 8 :

If B is an infinite subset of the countable set A, then B is countable.

Proof:

Let, $A = \{a_1, a_2, a_3, \ldots\}$

Given a is countable.

Let, n_1 be the ~~sub~~ smallest subscript for which $a_n \in B$

Let, n_2 be the next smallest and so-on,

Then $B = \{a_{n_1}, a_{n_2}, \ldots\}$

Thus elements of B can be labeled with 1, 2, 3 ....

Hence B is contable.

Theorem 9 :

The set of rational number's in $[0,1]$ is countable.

Proof :

The proof directly following the theorem 7, 8

Real Number's

Theorem 10 :

The set $[0,1]$ is uncountable

Proof :

Suppose $[0,1]$ were countable.

Then, $[0,1] = \{x_1, x_2, \ldots\}$ where every number in $[0,1]$ occure among the x_i Expending each x_i indicimal

We have,

$$x_1 = 0, a_1^1, a_2^1, a_3^1 \ldots\ldots$$

$$x_2 = 0. a_2^1, a_2^2, a_2^3 \ldots\ldots$$

$$\vdots$$

$$x_n = 0. a_n^1, a_n^2, a_n^3 \ldots\ldots$$

$$\vdots$$

Let b_1 be any integer from 0 to 8 $\ni b_1 \neq a_1^1$

Let b_2 be any integer from 0 to 8 $\ni b_2 \neq a_2^2$

In general for each $n = 1, 2 \ldots\ldots$, let b_n be any Integer from 0 to 8 $\ni b_n \neq a_n^n$

Let $y = 0. b_1, b_2, \ldots\ldots b_n \ldots$ then for any n.

the dicimal expansion for y ditters from x_n since $b_n \neq a_n^n$

More over the decimal expansion for y is unique since no b_n is equal to 9. Hence $y \neq x_n \; \forall n$ and $0 \leq y \leq 1$.

Which a $\Rightarrow\Leftarrow$ to our assumption, every number occurs among the x_i

Hence $[0,1]$ is uncountable.

Carollary : 11

The set $\mathbb{R}$ is uncountable.

Proof :-

Suppose $\mathbb{R}$ is countable. then by theorem 8, the subset of $\mathbb{R}$ $[0,1]$ is also countable

Which is $\Rightarrow\Leftarrow$ to the fact that $[0,1]$ is Uncountable, Hence $\mathbb{R}$ is Uncountable.

Cantor Set :

The cantor set is the set obtaind from $[0,1]$ by removing the sequence of open interval,

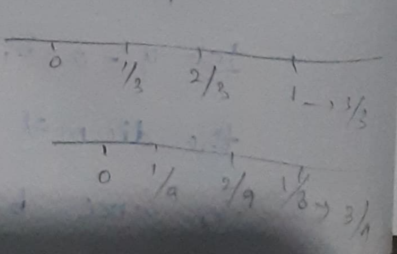
$\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{9}, \frac{2}{9}\right), \left(\frac{7}{9}, \frac{8}{9}\right)$ etc...

Thus the cantor set is the intersection of

$\{E_n \mid n \in \mathbb{N}\}$

Where each $E_i \in [0,1]$

Construction of Cantor set :-



$E_1 = [0,1]$

$E_2 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$

$$E_3 = \left[0, \tfrac{1}{9}\right] \cup \left[\tfrac{2}{9}, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, \tfrac{7}{9}\right] \cup \left[\tfrac{8}{9}, 1\right]$$

⋮

$$\text{Cantor set} = \bigcap_{n \in \mathbb{N}} E_n$$

Least Upper bound's :-

Definition :

⇒ The subset $A \subset \mathbb{R}$ is said to be bounded above if $\exists\, N \in \mathbb{R} \ni x \leq N \;\forall x \in A$.

⇒ The subset $A \subset \mathbb{R}$ is said to be bounded below if $\exists\, M \in \mathbb{R} \ni M \leq x \;\forall x \in A$.

⇒ A is bounded if A is bounded above.

⇒ A is bounded if and A is bounded below.

(ie) A is bounded $\Leftrightarrow A \subset [M, N]$ for some interval $[M, N]$ of finite length.

Paragraph :

⇒ The set I of ~~the~~ positive (+ive) integer's is bounded below but not bounded above.

⇒ Hence I is not bounded.

Example :

$[0, 1]$ is bounded above and bounded below.

Hence $[0, 1]$ is bounded.

Definition :- [Upper bound and Lower bound]

Upper bound :- [U.B]

⇒ If $A \subset R$ is bounded above, then N is called an upper bound, for A if $x \leq N \; \forall x \in A$.

Lower bound :- [L.B]

⇒ If $A \subset R$ is bounded below, then M is called a lower bound, for A if $M \leq x \; \forall x \in A$.

Least Upper bound and Greatist lower bound :-

Definition :-

Least Upper bound :- [L.U.B]

⇒ Let $A \subset R$, be bounded above. The number L is called least Upper bound for A if,

1] L is an Upper bound for A.

2] no number smaller then L is a Upper bound for A.

Greatist Lower bound :- [G.L.B]

⇒ Let $A \subset R$, be bounded below. The number l is called greatist lower bound for A if.

1] l is an lower bound for A.

2] no number greater than l is a lower bound for A.

examples:-

1] $B = (3, 4)$

G.L.B = 3

L.U.B = 4

2] $B = \{0\}$

G.L.B = 0

L.U.B = 0

3] $B = \left\{\frac{1}{2}, \frac{3}{4}, \ldots\ldots \frac{2^{n}-1}{2^{n}}, \ldots\right\}$

G.L.B = $\frac{1}{2}$

L.U.B = 1

4) $\{\pi+1, \pi+2, \pi+3, \ldots\}$

G.L.B = $\pi+1$

L.U.B = No least Upper bound

Definition - Least Upper bound axiom:

If A is any nonempty subset of R that is bounded above, then A has a least upper bound in R

Note:

The L.U.B axiom does not hold if R is replaced by set of all rational number's.

example:-

if $A = \{1, 1.4, 1.41, 1.414, \ldots\}$

then in R, L.U.B for A = $\sqrt{2}$

but $\sqrt{2}$ is not in the set of rationals.

Theorem:- 12

If A is any non empty subset of R that is bounded below, then A has greatest lower bound in R

Proof:

$\Rightarrow$ Let $B \subset R$ be the set of all $x \in R$ such that $(-x) \in A$

$\Rightarrow$ Since A is bounded below

there exists a $M \in R$ such that $M \leq x \; \forall \; x \in A$

$\Rightarrow -M \leq (-x), \ \forall x \in A$

$\Rightarrow -x \leq -M, \ \forall x \in A$

(ie) $y \leq -M, \ \forall y \in B$

$\Rightarrow -M$ is an Upper bound for B.

(ie) B is bounded above.

$\therefore$ By the Least Upper bound axiom,

B has a least Upper bound in R.

Proof that, If Q is a least upper bound for B.

then $-Q$ is greatest lower bound for A.

Let Q be the L.U.B for B.

(ie) 1) Q is an Upper bound for B.

2) no number ~~great~~ smaller than Q is an Upper bound for B.

$\Rightarrow$ 1) $-Q$ is a lower bound for A.

$\Rightarrow$ 2) No number greater than $-Q$ is a lower bound for A.

$\therefore -Q$ is the G.L.B for A.

A has the G.L.B in R.