Unit - III

Differentiation

Derivative is the limiting value of the change in the dependent variable divided by the change in the independent variable. The process of finding the derivatives is differentiation.

Let x and y be independent and dependent variables respectively and $y = f(x)$.

$$y + \Delta y = f(x + \Delta x)$$

$$\Delta y = f(x + \Delta x) - f(x)$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{Lt} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

This limits may exist.

$$\therefore \frac{dy}{dx} = \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{Lt} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$\frac{dy}{dx}$ is a symbol.

It does not mean any division

Derivative which is marginal rate of change.

1. Derivatives Of Standard functions from First principle

Derivative of x^n

Example 1: Find $\frac{d(x^n)}{dx}$ from principle.

Let $y = x^n$

$$y + \Delta y = (x + \Delta x)^n$$

$$\Delta y = (x + \Delta x)^n - x^n$$

$$\therefore \frac{dy}{dx} = \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{Lt} \frac{(x + \Delta x)^n - x^n}{\Delta x}$$

$$= \underset{x + \Delta x \to x}{Lt} \frac{(x + \Delta x)^n - x^n}{(x + \Delta x) - x} = n x^{n-1}$$

$$\frac{d(x^n)}{d(x)} = n x^{n-1}$$

Derivative of e^x

Example 2: Find $\frac{d(e^x)}{dx}$ from first principle.

Solution:

Let $y = e^x$

$$y + \Delta y = (e^{x + \Delta x})$$

$$= e^{x + \Delta x}$$

$$\Delta y = e^{x + \Delta x} - e^x$$

$$\therefore \frac{dy}{dx} = \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{Lt} \frac{e^{x + \Delta x} - e^x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{e^x (e^{\Delta x} - 1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x}$$

$$= e^x \, 1$$

$$\frac{d(e^x)}{dx} = e^x$$

Derivative of log x

Example 3: Prove that $\frac{d(\log_e x)}{dx} = \frac{1}{x}$

Let $y = \log_e x$

$$y + \Delta y = \log_e (x + \Delta x)$$

$$\Delta y = \log_e (x + \Delta x) - \log_e x$$

$$= \log_e \left(\frac{x + \Delta x}{x}\right)$$

$$= \log_e \left(1 + \frac{\Delta x}{x}\right)$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\log_e\left(1 + \frac{\Delta x}{x}\right)}{\Delta x}$$

$$= \frac{1}{x} \lim_{\frac{\Delta x}{x} \to 0} \frac{\log_e\left(1 + \frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}}$$

$$= \frac{1}{x} \cdot 1 = \frac{1}{x}$$

$$\frac{d(\log_e x)}{dx} = \frac{1}{x}$$

Derivative of a Constant

Example 4:

Prove that $\frac{d(K)}{dx} = 0$ where K is a constant.

Solution

Let $y = K$

$$y + \Delta y = K$$

$$\Delta y = K - K = 0$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0$$

$$\frac{d(K)}{d(x)} = 0$$

2. CERTAIN RULES OF DIFFERENTIATION:

Rule for a Constant times a function.

Example 5:

Prove that $\frac{d}{dx}(Ku) = K\frac{du}{dx}$ where K is a constant and u is a function of x.

Let $y = Ku$

$$y + \Delta y = K(u + \Delta u)$$

$$\Delta y = K(u + \Delta u) - Ku$$

$$= K\Delta u$$

$$\therefore \frac{dy}{dx} = \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{Lt} \left(K \frac{\Delta u}{\Delta x} \right)$$

$$\frac{dy}{dx} = \underset{\Delta x \to 0}{Lt} K . \underset{\Delta x \to 0}{Lt} \frac{\Delta u}{\Delta x}$$

$$= K \frac{du}{dx}$$

$$\therefore \frac{d}{dx}(Ku) = K \frac{du}{dx}$$

1) The Addition rule and the difference rule

Example 6:

Prove that $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$ where u and v are functions of x.

Let $y = u + v$

$$y + \Delta y = (u + \Delta u) + (v + \Delta v)$$

$$\Delta y = u + \Delta u + v + \Delta v - (u + v)$$

$$= \Delta u + \Delta v$$

$$\therefore \frac{dy}{dx} = \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{Lt} \left(\frac{\Delta u + \Delta v}{\Delta x} \right)$$

$$= \underset{\Delta x \to 0}{Lt} \frac{\Delta u}{\Delta x} + \underset{\Delta x \to 0}{Lt} \frac{\Delta v}{\Delta x}$$

$$= \frac{du}{dx} + \frac{dv}{dx}$$

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

2. Difference rule:

$Y = U - V,$

$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$ ie $\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$

Example: $y = x^9 + x$

$$\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$$

$$\underset{u}{x^9} + \underset{v}{x}$$

$$\frac{dy}{dx} = \frac{d(x^9)}{dx} + \frac{d(x)}{dx}$$

$$= 9x^8 + 1$$

3. The Product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Example 7. Prove that $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ when u and v are functions of x.

Let $y = uv$

$$y + \Delta y = (u + \Delta u)(v + \Delta v)$$

$$y + \Delta y = uv + u'\Delta v + v\Delta u + \Delta u \Delta v$$

$$\Delta y = u\Delta v + v\Delta u + \Delta u \Delta v$$

$$\therefore \frac{dy}{dx} = \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{Lt} \left(u \frac{\Delta v}{\Delta x} + v \frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x} \Delta v \right)$$

$$= u \underset{\Delta x \to 0}{Lt} \frac{\Delta v}{\Delta x} + v \underset{\Delta x \to 0}{Lt} \frac{\Delta u}{\Delta x} + \underset{\Delta x \to 0}{Lt} \frac{\Delta u}{\Delta x} \underset{\Delta x \to 0}{Lt} \Delta v$$

$$= u \frac{dv}{dx} + v \frac{du}{dx} + \frac{du}{dx} \times 0$$

$$\therefore \frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

The Quotient rule

Example 8:

Prove that $\frac{d}{dx}\left(\frac{u}{v}\right) = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

where u and v are functions of x.

$$y = \frac{u}{v}$$

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}$$

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}$$

$$= \frac{v(u + \Delta u) - u(v + \Delta v)}{v(v + \Delta v)}$$

$$= \frac{uv + v\Delta u - uv - u\Delta v}{v(v + \Delta v)}$$

$$= \frac{v\Delta u - u\Delta v}{v^2 + v\Delta v}$$

$$\therefore \frac{dy}{dx} = \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{Lt} \frac{(v\Delta u - u\Delta v)\Delta x}{v^2 + v\Delta v}$$

$$= \frac{v \underset{\Delta x \to 0}{Lt} \frac{\Delta u}{\Delta x} - u \underset{\Delta x \to 0}{Lt} \frac{\Delta v}{\Delta x}}{v^2 + v \underset{\Delta x \to 0}{Lt} \Delta v}$$

$$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

examples

$$y = \frac{e^x}{x^2} \quad \begin{matrix} - u \\ - v \end{matrix}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{x^2\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{x^2e^x - e^x(2x)}{x^4}$$

$$= \frac{xe^x(x-2)}{x^4}$$

$$= \frac{(x-2)e^x}{x^3}$$

Some More Problems:

Example 9: Find $\frac{dy}{dx}$ if $y = 5x^3 + 9x^2$

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

$$\frac{dy}{dx} = 5\frac{d}{dx}(x^3) + 9\frac{d}{dx}(x^2)$$

$$= 5(3x^2) + 9(2x)$$

$$\frac{dy}{dx} = 15x^2 + 18x$$

Example 10. Find the differential coefficient (derivative) of the function $y = x^2 - 4$ with respect to x.

$$\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) - \frac{d}{dx}(4)$$

$$= 2x - 0$$

$$\frac{dy}{dx} = 2x$$

Example 15.

i) $y = (x^2 + 5)(3x + 1)$

$\quad\quad\quad\quad u \quad\quad\quad v$

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= (x^2+5)\frac{d}{dx}(3x+1) + (3x+1)\frac{d}{dx}(x^2+5)$$

$$= (x^2+5)\,3 + (3x+1)(2x)$$

$$= 3x^2 + 15 + 6x^2 + 2x$$

$$= 9x^2 + 2x - 15$$

Another

$$\frac{d}{dx}(uv) = uv' + vu'$$

$u = x^2 + 5$

$u' = 2x + 0$

$= 2x$

$v = 3x + 1$

$v' = 3 + 0$

$= 3$

$$\frac{d}{dx}\left[(x^2+5)(3x+1)\right] = (x^2+5)(3) + (3x+1)(2x)$$

$$= (3x^2+15) + (6x^2 + 2x)$$

$$= 9x^2 + 2x + 15$$

ii) $y = \dfrac{3x^2}{4x-1}$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$u = 3x^2$

$u' = 6x$

$v = 4x - 1$

$v' = 4(1)0$

$v' = 4$

$$\frac{d}{dx}\left(\frac{3x^2}{4x-1}\right) = \frac{(4x-1)(6x) - (3x^2)(4)}{(4x-1)^2}$$

$$= \frac{(24x^2 - 6x) - (12x^2)}{(4x-1)^2}$$

$$\frac{dy}{dx} = \frac{(12x^2 - 6x)}{(4x-1)^2}$$

Example 17.

$$y = \frac{7e^x \log_e x}{3x-5}$$

$$d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$\frac{dy}{dx} = \frac{(3x-5)\frac{d}{dx}(7e^x \log x) - (7e^x \log x)\frac{d}{dx}(3x-5)}{(3x-5)^2}$$

$$d(uv) = uv' + vu'$$

$$u = e^x \qquad v = \log x$$

$$\frac{dy}{dx} = \frac{7(3x-5)(e^x \frac{1}{x} + \log x e^x) - 7e^x \log x(3-0)}{(3x-5)^2}$$

$$= \frac{7e^x(3x-5)\left(\frac{1}{x} + \log x\right) - 7e^x \log x(3)}{(3x-5)^2}$$

$$= \frac{7e^x\left(3 - \frac{5}{x} - 3x\log x - 5\log x - 3\log x\right)}{(3x-5)^2}$$

$$= \frac{7e^x\left(3 - \frac{5}{x} + 3x\log x - 8\log x\right)}{(3x-5)^2}$$

Example 11 :

Differentitate the following with respect to x.

i) $x^3 - 3x^2 + 4x + 3$

$$\frac{d}{dx}(x^3 - 3x^2 + 4x + 3) = \frac{d}{dx}(x^3) - \frac{3d}{dx}(x^2) + \frac{4d}{dx}(x) + \frac{d}{dx}(3)$$

$$= 3x^2 - 3(2x) + 4(1) + 0$$

$$= 3x^2 - 6x + 4$$

$$\frac{d}{dx} = 3x^2 - 6x + 4$$

ii) $x^5 + 3\log x - 4e^x$

$$\frac{d}{dx}(x^5 + 3\log x - 4e^x) = \frac{d}{dx}(x^5) + \frac{3d}{dx}(\log x) - \frac{4d}{dx}(e^x)$$

$$= 5x + \frac{3}{x} - 4e^x$$

iii) $4\sqrt{x} + \frac{3}{x} + 5e^x$

$$\frac{d}{dx}\left(4\sqrt{x} + \frac{3}{x} + 5e^x\right) = \frac{4d}{dx}(\sqrt{x}) + \frac{3d}{dx}\left(\frac{1}{x}\right) + \frac{5d}{dx}(e^x)$$

$$= \frac{4}{2\sqrt{x}} - \frac{3}{x^2} + 5e^x\left(\frac{1}{x}\right)$$

$$= x^{-1} \text{ and } \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= -x^{-2} = -\frac{1}{x^2}$$

$$= \frac{2}{\sqrt{x}} - \frac{3}{x^2} - 5e^x$$

Exercises

3. Find the derivatives of

i) x^5

$$\frac{d(x^5)}{dx} = 5x^{5-1}$$
$$= 5x^4$$

ii) $\sqrt{x}$

$$\frac{d(x^{\frac{1}{2}})}{dx} = \frac{1}{2}x^{\frac{1}{2}-1}$$
$$= \frac{1}{2}x^{-\frac{1}{2}}$$
$$= \frac{1}{2\sqrt{x}}$$

iii) $\frac{1}{x^2}$

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{d}{dx}(x^{-2})$$
$$= -2x^{-2-1}$$
$$= -2x^{-3}$$
$$= \frac{-2}{x^3}$$

4. Find the derivatives of

i) $y = 7x^4 + 2x^3 + 4x^2 - 6x + 100$

$$\frac{d}{dx} = 28x^3 + 6x^2 + 8x - 6$$

ii) $2x^{3/2} - 3\log_e x + 6$

$$\frac{dy}{dx} = \cancel{2}\left(\frac{3}{\cancel{2}}\right)x^{3/2-1} - 3\left(\frac{1}{x}\right) + 0$$

$$= 3x^{\frac{1}{2}} - \frac{3}{x}$$

$$= 3\sqrt{x} - \frac{3}{x}$$

iii) $x^4 + 2e^x + 3$

$y = $ Lt $\quad y = x^4 + 2e^x + 3$

$$\frac{dy}{dx} = 4x^3 + 2e^x$$

iv) $y = (2x+3)(3x^2)$

$$\frac{dy}{dx} = (2x+3)(6x) + 3x^2(2)$$

$$= 6x\,(2x+3) + 6x^2$$

$$= 6x[2x + 3 + x]$$

$$= 6x[3x + 3]$$

$$= 18x[x + 1]$$

v) $y = x^2 e^x$

$$\frac{dy}{dx} = x^2\frac{de^x}{dx} + e^x\frac{d}{dx}x^2$$

$$= x^2e^x + e^x(2x)$$

$$= e^x[x^2 + 2x]$$

$$= xe^x(x+2)$$

vi) $x^2 \log x$

$$y = x^2 \log x$$

$$\frac{dy}{dx} = x^2 \frac{1}{\cancel{x}} + \log x (2x)$$

$$= x(2 \log x + 1)$$

vii) $\frac{\log x}{x^2}$

$$y = \frac{\log x}{x^2}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{x^2 \frac{d}{dx} \log x - \log x \frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{x^2 \frac{1}{\cancel{x}} - \log x (2x)}{x^4}$$

$$= \frac{\cancel{x}[1 - 2 \log x]}{x^{\cancel{4}\,3}}$$

$$= \frac{1 - 2 \log x}{x^3}$$

viii) $(4x-8)(3x+11)$

$$\frac{dy}{dx} = \frac{(3x+11)(4)-(4x-8)(3)}{(3x+11)^2}$$

$$= \frac{12x+44-12x+24}{(3x+11)^2}$$

$$= \frac{68}{(3x+11)^2}$$

ix) $x^2/(x^3+1)$

$$y = x^2/(x^3+1)$$

$$\frac{dy}{dx} = \frac{(x^3+1)(2x) - x^2(3x^2)}{(x^3+1)^2}$$

$$= \frac{2x^4+2x-3x^4}{(x^3+1)^2}$$

$$= \frac{x(2-x^3)}{(x^3+1)^2}$$

x) $2x/(2x^2-3)$

$$\frac{dy}{dx} = \frac{(2x^2-3)(2) - 2x(4x)}{(2x^2-3)^2}$$

$$= \frac{4x^2-6-8x^2}{(2x^2-3)^2}$$

$$= \frac{-4x^2-6}{(2x^2-3)^2}$$

$$= \frac{-2(2x^2+3)}{(2x^2-3)^2}$$

xi) $y = \frac{2x+1}{4x+5}$

$$\frac{dy}{dx} = \frac{(4x+5)(2)-(2x+1)(4)}{(4x+5)^2}$$

$$= \frac{8x+10-8x-4}{(4x+5)^2}$$

$$= \frac{6}{(4x+5)^2}$$

xii) $\frac{5}{1-3x}$

$$y = \frac{5}{1-3x}$$

$$\frac{dy}{dx} = \frac{(1-3x)(0)-5(-3)}{(1-3x)^2}$$

$$= \frac{15}{(1-3x)^2}$$

xiii) e^{2+5x}

$$y = e^{2+5x}$$

$$\frac{dy}{dx} = e^{5x+2}\frac{d}{dx}(5x+2)$$

$$= 5e^{5x+2}$$

xiv) $(1-5x)^6$

$$y = (1-5x)^6$$

$$\frac{dy}{dx} = 6(1-5x)^5\frac{d}{dx}(1-5x)$$

$$= 6(1-5x)^5(-5)$$

$$= -30(1-5x)^5$$