CHAPTER II

RELATIONS

§ 1. CARTESIAN PRODUCT OF TWO SETS

Let A and B be two sets. Consider the pair (a, b) in which the first element a is from A and the second element b is from B. Then (a, b) is called an ***ordered pair***. In an ordered pair, the order in which the two elements are written is important. Thus (a, b) and (b, a) are ***different ordered pairs***. We say that two ordered pairs (a, b) and (c, d) are equal if and only if a = c and b = d. Now we can define the Cartesian product of two sets.

Definition

Let A and B be two sets. Then the set of all ordered pairs (x, y) where $x \in A$ and $y \in B$ is called the ***Cartesian Product*** of the sets A and B and is denoted by $A \times B$, read as "Cartesian Product of A and B" or simply "A cross B". Thus we have

$$A \times B = \{ (x, y) : x \in A \text{ and } y \in B \}$$

Illustration

Let A = { 1, 2, 3, 4 } and B = { 2, 3, 7 }

$$\text{Then } A \times B = \left\{ \begin{matrix} (1, 2), (1, 3), (1, 7), (2, 2), (2, 3), (2, 7) \\ (3, 2), (3, 3), (3, 7), (4, 2), (4, 3), (4, 7) \end{matrix} \right\}$$

$$B \times A = \left\{ \begin{matrix} (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2) \\ (3, 3), (3, 4), (7, 1), (7, 2), (7, 3), (7, 4) \end{matrix} \right\}$$

It is to be noted that $A \times B = B \times A$, if the sets A and B are different.

Note

1. If A has m elements and B has n elements, then $A \times B$ will have mn elements, and $B \times A$ will also have mn elements. But the elements of $B \times A$ need not be the same as the elements of $A \times B$.
2. If either A or B is the null set, then $A \times B$ is the null set.
3. We can also define the Cartesian Product of more than two sets in a similar way. Thus, if $A_1, A_2, \ldots, A_n$ are n sets, their Cartesian Product is defined as

$$A_1 \times A_2 \times \ldots \times A_n = \{ (a_1, a_2, \ldots, a_n) : a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n \}$$

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$$A \times B = \{ (x, y) : x \in A \text{ and } y \in B \}$$

Illustration

Let A = { 1, 2, 3, 4 } and B = { 2, 3, 7 }.

$$\text{Then } A \times B = \left\{ \begin{array}{l} (1, 2), (1, 3), (1, 7), (2, 2), (2, 3), (2, 7) \\ (3, 2), (3, 3), (3, 7), (4, 2), (4, 3), (4, 7) \end{array} \right\}$$

$$B \times A = \left\{ \begin{array}{l} (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2) \\ (3, 3), (3, 4), (7, 1), (7, 2), (7, 3), (7, 4) \end{array} \right\}$$

It is to be noted that $A \times B \neq B \times A$, if the sets A and B are different.

Note

1. If A has m elements and B has n elements, then $A \times B$ will have mn elements, and $B \times A$ will also have mn elements. But the elements of $B \times A$ need not be the same as the elements of $A \times B$.
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$$A_1 \times A_2 \times \ldots \times A_n = \{ (a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n \}$$

For example, if R is the set of all real numbers, then

$R \times R = \{ (a_1, a_2) \mid a_1 \in R, a_2 \in R \}$ and this is written as R^2.

$R \times R \times R = \{ (a_1, a_2, a_3) \mid a_1 \in R, a_2 \in R, a_3 \in R \}$ and this is written as R^3

4. R^2 represents the two dimensional space. An element (a, b) in R^2 represents the point (a, b) in the plane whose x coordinate is a and y coordinate is b. Similarly R^3 represents the three dimensional space.

Worked Examples

W.E.1. Suppose in R × R, *the ordered pairs* (x − 2, 2y+1) *and* (y − 1, x+2) *are equal. Find x and y.*

Solution

By the definition of the equality of ordered pairs, we have

$x - 2 = y - 1$ i.e., $x - y = 1$ ....(1)

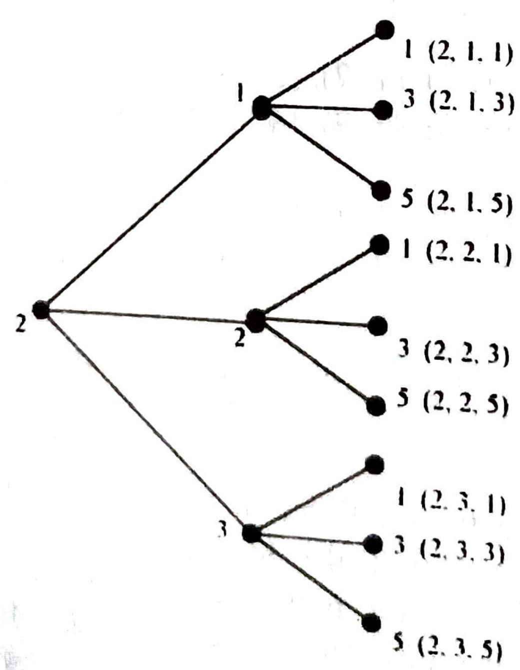
and $2y + 1 = x + 2$ i.e., $x - 2y = -1$ ....(2)

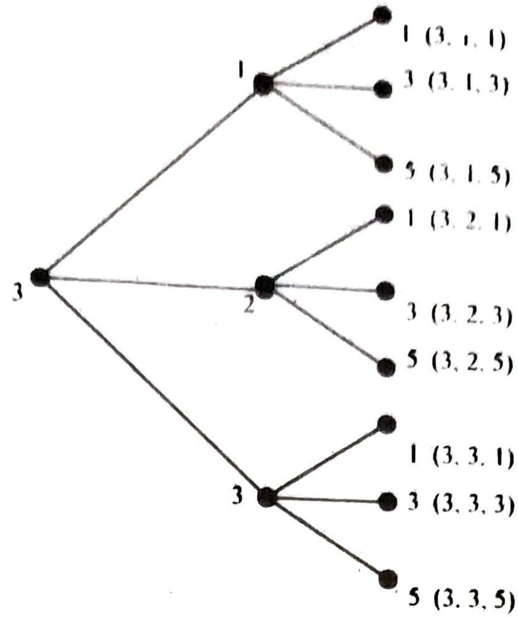
Solving (1) and (2), we get x = 3, y = 2.

W.E. 2. If A = { 2, 3 }, B = { 1, 2, 3} *and* C = { 1, 3, 5 }, *find* A × B × C.

Solution

A convenient method of finding A × B × C is through the diagram shown below:





$$A \times B \times C = \left\{ \begin{array}{l} (2,1,1),(2,1,3),(2,1,5),(2,2,1),(2,2,3),(2,2,5), \\ (2,3,1),(2,3,3),(2,3,5),(3,1,1),(3,1,3),(3,1,5), \\ (3,2,1),(3,2,3),(3,2,5),(3,3,1),(3,3,3),(3,3,5) \end{array} \right\}$$ •

W.E.3. *If* $A = \{c, d\}, B = \{1, 2\}, C = \{2, 3\}$, *find*

i) $A \times (B \cup C)$ ii) $(A \times B) \cup (A \times C)$

iii) $A \times (B \cap C)$ iv) $(A \times B) \cap (A \times C)$

Solution

i) First $(B \cup C) = \{1, 2, 3\}$

Then $A \times (B \cup C) = \{c, d\} \times \{1, 2, 3\}$

$= \{(c, 1), (c, 2), (c, 3), (d, 1), (d, 2), (d, 3)\}$.

ii) First $A \times B = \{(c, 1), (c, 2), (d, 1), (d, 2)\}$

$A \times C = \{(c, 2), (c, 3), (d, 2), (d, 3)\}$

Then $(A \times B) \cup (A \times C) = \{(c, 1), (c, 2), (d, 1), (d, 2), (c, 3), (d, 3)\}$

We find from (i) and (ii), that

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

iii) First $B \cap C = \{2\}$

Then $A \times (B \cap C) = \{c, d\} \times \{2\} = \{(c, 2), (d, 2)\}$

iv) $(A \times B) \cap (A \times C) = \{(c, 2), (d, 2)\}$.

We find from (iii) and (iv) that

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$ •

W.E.4. *If A, B, C are any three sets, prove that* $A \times (B \cup C) = (A \times B) \cup (A \times C)$ *(M.C.A., Nov '98, M.U.)*

Proof

a) Let us show that

$$A \times (B \cup C) \subset (A \times B) \cup (A \times C)$$

Let (x, y) be any element of $A \times (B \cup C)$

Then $x \in A$ and $y \in B \cup C$.

i.e., $x \in A$ and $y \in B$ or $x \in A$ and $y \in C$.

$\therefore (x, y) \in (A \times B)$ or $(x, y) \in (A \times C)$

i.e., $(x, y) \in (A \times B) \cup (A \times C)$

$$\therefore A \times (B \cup C) \subset (A \times B) \cup (A \times C) \quad \text{...........(1)}$$

b) Let us show that

$$(A \times B) \cup (A \times C) \subset A \times (B \cup C)$$

Let (z, w) be any element of $(A \times B) \cup (A \times C)$.

Then $(z, w) \in A \times B$ or $(z, w) \in A \times C$

i.e., $z \in A$ and $w \in B$ or $z \in A$ and $w \in C$

i.e., $z \in A$ and $w \in B$ or C

i.e., $(z, w) \in A \times (B \cup C)$

$$\therefore (A \times B) \cup (A \times C) \subset A \times (B \cup C) \quad \text{.............(2)}$$

From (1) and (2), we have

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

Exercises

1. If $B = \{-1, 1\}$ and $C = \{(1, -1), (1, 1), (-1, 1), (-1, -1)\}$, find A so that $A \times B = C$.
2. If $A \times B = \{(6, 2), (6, 3), (6, 4), (1, 2), (1, 3), (1, 4)\}$, find A and B.
3. Determine x or y so that the ordered pairs are equal in the following :
   a) $(4, x) = (4, 3)$ b) $(3y, a) = (12, a)$
   c) $(3x+1, 4) = (7, 4)$ d) (PASCAL, ALGOL) = (x, ALGOL)
4. Solve each of the following for x and y :
   a) $(5x, y+2) = (x-8, -y)$
   b) $(y^2, 4) = (2y-1, x^2)$
   c) $(y-2, 3-x) = (2, -3)$
5. A marketing research firm classifies a person according to the following two criteria :
   Sex : m = male f = female
   Highest level of education completed :
   e = elementary h = high school c = college g = graduate school
   a) How many categories are there in this classification scheme?
   b) List all the categories.

6. A medical experiment classifies each subject according to two criteria.
   Smoking Pattern :
   s = smoker n = nonsmoker
   Weight :
   u = under weight a = average weight
   o = over weight
   List all possible classifications in this scheme.
7. If $A = \{ p, q, r \}$, $B = \{ 2, 3 \}$ and $C = \{ a, b \}$, list all the elements of $A \times B \times C$.
8. Let $A = \{ a, b \}$ and $B = \{ 1, 2, 3 \}$. List the elements of
   i) $A \times B$ ii) $B \times A$ iii) $A \times A$ iv) $B \times B$
9. If $A = \{ a, b, c \}$ and $B = \{ b, c, d \}$, describe the sets
   i) $A \times A$ ii) $B \times B$
   iii) $B \times A$ iv) $A \times B$
   v) $(A \times A) \cap (B \times B)$ vi) $(A \times A) \cap (A \times B)$
   vii) $(B \times B) \cap (A \times B)$ viii) $(B \times A) \cap (A \times B)$
10. If A, B, C be any three sets, prove that
    $A \times (B \cap C) = (A \times B) \cap (A \times C)$ (M.C.A.,Dec '96, Bharathiar Uni.)
11. Let $A = \{ a, b \}$, $B = \{ 1, 2, 3, 4, 5 \}$ and $C = \{ 3, 5, 7, 9 \}$. Find $(A \times B) \cap (A \times C)$ without finding $A \times B$ and $A \times C$.
12. If A, B, C be any three sets, prove that
    i) $(A \cup B) \times C = (A \times C) \cup (B \times C)$
    ii) $(A \cap B) \times C = (A \times C) \cap (B \times C)$
13. Fill in the blanks :
    i) If $(x, y) \in A \times (B \cup C)$, then
    $x \in$ .... and $y \in$ ....
    ii) If $(x, y) \in (A \times B) \cup (A \times C)$, then
    a) $x \in$ .... b) $y \in$ .... or $y \in$ ....
    iii) If A_1 has n_1 elements, A_2 has n_2 elements and A_3 has n_3 elements, then $A_1 \times A_2 \times A_3$ has .... elements.
14. State extended rule of products. (B.E., Nov '96, M.S.U.)
15. If $A = (+, -)$ and $B = \{ 00, 01, 10, 11 \}$
    i) list all the elements of $A \times B$ (Oct '98, B.E., M.U.)
    ii) how many elements do A^4 and $(A \times B)^3$ have ?
    (Dec '96, M.C.A., Bharathiar Uni.)

Answers

1. $\{ 1, -1 \}$.
2. $A = \{ 6, 1 \}$ and $B = \{ 2, 3, 4 \}$
3. a) $x = 3$ b) $y = 4$ c) $x = 2$ d) x = PASCAL

4. a) $x = -2,\ y = -1$ b) $x = \pm 2,\ y = 1$ c) $x = 6,\ y = 4$.
5. a) 8 b) { (m, e), (m, h), (m, c), (m, g), (f, e), (f, h), (f, c), (f, g) }.
6. { (s, u), (s, a), (s, o), (n, u), (n, a), (n, o) }.
7. { (p, 2, a), (p, 2, b), (p, 3, a), (p, 3, b), (q, 2, a), (q, 2, b)
(q, 3, a), (q, 3, b), (r, 2, a), (r, 2, b), (r, 3, a), (r, 3, b) }
8. a) { (a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3) }
b) { (1, a), (1, b), (2, a), (2, b), (3, a), (3, b) }
c) { (a, a), (a, b), (b, a), (b, b) }
d) { (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3) }
9. i) { (a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c) }
ii) { (b, b), (b, c), (b, d), (c, b), (c, c), (c, d), (d, b), (d, c), (d, d) }
iii) { (b, a), (b, b), (b, c), (c, a), (c, b), (c, c), (d, a), (d, b), (d, c) }
iv) { (a, b), (a, c), (a, d), (b, b), (b, c), (b, d), (c, b), (c, c), (c, d) }
v) { (b, b), (b, c), (c, b), (c, c) }
vi) { (a, b), (a,c), (b, b), (b, c), (c, b), (c, c) }
vii){ (b, b), (b, c), (b, d), (c, b), (c, c), (c, d) }
viii) { (b, b), (b, c), (c, b), (c, c) }
11. { (a, 3), (a, 5), (b, 3), (b,5) }
13. i) $x \in A$ and $y \in B \cup C$
ii) a) $x \in A$ b) $y \in B$ or $y \in C$.
iii) $n_1 n_2 n_3$
15. i) { (+, 00), (+, 01), (+, 10), (−, 00), (−, 01), (−, 10), (−, 11) }.
ii) 16; 512.

§ 2. RELATIONS

The idea of a relation between the elements of two sets or between the elements of a set is quite common. For example, in the set H of all human beings, we have relations like 'father of', 'sister of', 'taller than' etc,. In the set N of natural numbers, we have relations like 'less than or equal to', 'a divisor of', 'square of'. Suppose we want to use symbol R to denote a relation such as 'father of', then we can write x R y if x is the father of y; or we can say that the ordered pair (x, y) is related by the relation R. Thus the set of all ordered pairs (x, y) satisfying x R y defines the relation R completely. So a relation gives rise to a subset of A × B if the relation is between an element of A and an element of B. This leads to a mathematical definition of a relation.

Definition

Let A and B be non empty sets. A relation R from the set A to the set B is a subset of $A \times B$.

Note This definition allows the empty subset of $A \times B$ to be a relation. This relation does not contain any ordered pair and is referred to as the null relation or the empty relation in $A \times B$. It also allows $A \times B$ as a relation which is referred to as the universal relation from A to B.

If B = A, we often say that the relation is on A.

If R is a subset of $A \times B$ and $(a, b) \in R$, we say that a is related to b by R and we also write a R b. If a is not related to b, we write $a \not R b$.

We now give a number of examples :

Examples

1. Let A = { 1, 2, 3 } and B = { a, b }. Then the subset R = { (1, a), (1, b), (2, a), (3, b) } is a relation from A to B defined by the following statements :

   1 R a, 1R b, 2 R a, 3 R b.
2. Let A and B be sets of real numbers. We define a relation R from A to B by a R b if and only if a = b.
3. Let A = { 1, 2, 3, 4 }. We define a relation R on A by a R b if and only if a < b. Then R = { (1, 2),(1, 3), (1,4), (2, 3), (2, 4), (3, 4) }.
4. Let $A = Z^+$, the set of all positive integers. We define the relation R "is a divisor of" in A by a R b if and only if a divides b. Note that 4 R 12 but $8 \not R 18$.
5. Let A be the set of all real numbers. We define the relation x R y if and only if x and y satisfy the inequality

$$\frac{x^2}{9} + \frac{y^2}{4} \leq 1$$

Then the set R consists of all points on and in the interior of the ellipse in figure 1.

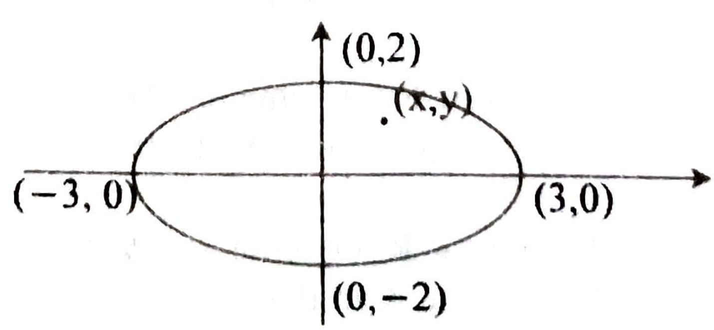


Figure 1. $\{(x,y): \frac{x^2}{9} + \frac{y^2}{4} \leq 1 \}$

Worked Examples

W.E.1. A manufacturer of automobiles has five factories $\{a_1, a_2, a_3, a_4, a_5\}$ *and six distribution centres* $\{b_1, b_2, b_3, b_4, b_5, b_6\}$. *The following table gives the distance (in Kms.) from* a_i *to* b_j

	b_1	b_2	b_3	b_4	b_5	b_6
a_1	1200	1100	400	600	1800	700
a_2	800	700	1200	450	400	500
a_3	1000	600	1000	650	600	600
a_4	250	400	500	350	900	600
a_5	800	280	300	400	1300	2400

We define the relation $a_i \, R \, b_j$, *if and only if the distance from* a_i *to* b_j *is at least* 800 *Kms. List the elements in R.*

Solution

$$R = \left\{ \begin{array}{l} (a_1, b_1), (a_1, b_2), (a_1, b_5), (a_2, b_1), (a_2, b_3), (a_3, b_1), \\ (a_3, b_3), (a_4, b_5), (a_5, b_1), (a_5, b_5), (a_5, b_6) \end{array} \right\}$$

W.E.2. Sets A *and* B *have respectively* m *and* n *elements. How many elements has* $A \times B$? *How many different relations are there from* A *to* B ?

Solution

The set $A \times B$ has mn elements.

Each subset of $A \times B$ is a relation from A to B.

As the number of subsets of $A \times B$ is 2^{mn}, there are 2^{mn} different relations from A to B.

[We know that a set of n elements has 2^n subsets. See Chapter 1, theorem 2.]

DOMAIN AND RANGE OF A RELATION

Let R be a relation from A to B i.e., let R be a subset of $A \times B$. The domain D of the relation R is the set of all the first elements of the ordered pairs which belong to R.

$$D = \{ a \mid a \in A \text{ and } (a, b) \in R \text{ for some } b \in B \}$$

The range E of the relation is the set of all the second elements of ordered pairs in R.

$$\text{i.e., } E = \{ b \mid b \in B, (a, b) \in R \text{ for some } a \in A. \}$$

Clearly the domain of a relation from A to B is a subset of A and its range is a subset of B.

Example. Let A = { 1, 2, 3, 4 } and B = { r, s, t } and R = { (1, r),(2, s), (3, r)}. The domain of R is the set { 1, 2, 3 } and the range of R is the set { r, s }.

Exercises

1. Let R be the relation from A = { 1, 3, 5, 7, 9 } to B = { 2, 4, 6, 8 } which is defined as a R b if and only if $a > b$. List the elements of R and find its domain and range.
2. Let R be the relation from A = { 2, 3, 4, 5 } to B = { 3, 6, 7, 10 }, which is defined by the expression "x divides y".
   i) Write R as a set of ordered pairs.
   ii) Find its range and domain.
3. Let R be the relation in A = { 2, 3, 4, 5, 6 } defined by the x R y if and only if $| x - y |$ is divisible by 3. Write R as subset of $A \times A$.
4. Let R be the relation on the set N of all natural numbers given by the expression $x + 3y = 12$.
   i.e., $R = \{ (x, y) \mid x \in N, y \in N, x + 3y = 12 \}$.
   i) Write R as a set of ordered pairs.
   ii) Find the domain and range of R.
5. Let R be the relation on the set N of all natural numbers given by the expression $2x + 4y = 15$.
   i) Write R as a set of ordered pairs.
   ii) Find the domain and range of R.
6. A relation R in Z, the set of all integers is defined as follows :
   $x R y \Leftrightarrow x$ is the square of y. Which of the following statements are true ?
   i) 4 R 2 ii) 2 R 4
   iii) 9 R (−3) iv) 9 R 3 v) 3 R 9.
7. A relation R in C, the set of all complex numbers is defined as follows :
   $x R y \Leftrightarrow x$ is the conjugate of y.
   Which of the following statements are true ?
   i) 4 R 4 ii) i R i
   iii) −2 R 2 iv) (1+i) R (1−i)
   v) (1−i) R (1+i) vi) (−1+i) R (−1−i)
8. Given that R is a relation in { 1, 2, 3, 4, 5 } such that 1 R 3, 2 R 4, 3 R 5, what are the domain and the range of R ? Give some verbal description of the relation.
9. If S = { 1, 2, 3 } and T = { x, y }, list all the elements of $S \times T$.
10. Let A = { 1, 2, 3, 4 } and B = { 1, 2, 3, 4 }. In each of the following, find all the pairs of $A \times B$ that belong to R.
    a) $R = \{ (x, y) \mid x \geq y \}$ b) $R = \{ (x, y) \mid x > y \}$
    c) $R = \{ (x, y) \mid x \leq y \}$ d) $R = \{ (x, y) \mid x = y^2 \}$

Answers

1. $R = \{ (3, 2), (5, 2), (5, 4), (7, 2), (7, 4), (7, 6), (9, 2), (9, 4), (9, 6), (9, 8) \}$
   Domain(R) = { 3, 5, 7, 9 }   Range(R) = { 2, 4, 6, 8 }
2. $R = \{ (2, 6), (2, 10), (3, 3), (3, 6), (5, 10) \}$.
   Domain(R) = { 2, 3, 5 }   Range(R) = { 6, 10, 3 }
3. $R = \{ (2, 2), (2, 5), (3, 3), (3, 6), (4, 4), (5, 5), (5, 2), (6, 3), (6, 6) \}$
4. $R = \{(3, 3), (6, 2), (9, 1) \}$; Domain(R) = {3, 6, 9 }; Range(R) = {3, 2, 1}
5. i) ϕ ii) domain is ϕ and range is ϕ.
6. (i), (iii), (iv) are true.
7. (i), (iv), (v), (vi) are true.
8. Domain = { 1, 2, 3 }; Range = { 3, 4, 5 }; $a + 2 = b$ where $a, b \in R$.
9. $S \times T = \{ (1, x), (1, y), (2, x), (2, y), (3, x), (3, y) \}$.
10. a) { (1, 1), (2, 1), (2, 2) (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3), (4, 4) }
    b) { (2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3) }
    c) {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4) }
    d) { (1, 1), (4, 2) }.

§ 3. REPRESENTATION OF RELATION

I. MATRIX OF A RELATION

If $A = \{ a_1, a_2, \ldots, a_m \}$ and $B = \{ b_1, b_2, \ldots, b_n \}$ are finite sets containing m and n elements respectively and R is a relation from A to B, we can represent R by the $m \times n$ matrix

$M_R = [m_{ij}]$ which is defined as follows :

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R. \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$$

Example. Let $A = \{ a_1, a_2, a_3 \}$ and $B = \{ b_1, b_2, b_3, b_4 \}$ and R be the relation given by $R = \{ (a_1, b_1), (a_1, b_4), (a_2, b_2), (a_2, b_3), (a_3, b_1), (a_3, b_3) \}$.
Clearly the matrix of R is

$$M_R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

II. Digraph Of A Relation

The relation R, on a finite set A, can be represented pictorially as follows : A small circle is drawn for each element of A and marked with the corresponding element. These circles are called vertices. An arc is drawn from the vertex a_i to the vertex a_j, if and only if a_i R a_j. This is called an edge. This pictorial representation of R is called a directed graph or digraph of R.

Thus, if R is a relation on A, the vertices in the digraph of R correspond exactly to the elements of the set A and the edges correspond exactly to the ordered pairs in R.

In a digraph of R, the indegree of a vertex is the number of edges terminating at the vertex. The outdegree of a vertex is the number of edges leaving the vertex.

Example 1. Let A = { a, b, c, d } and R the relation on A that has the matrix

$$M_R = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Construct the digraph of R and list the indegrees and outdegress of all vertices.
Clearly R = { (a, a), (a, b), (a, d), (b, c), (c, c), (c, d), (d, a) }.
The digraph of R is shown below.

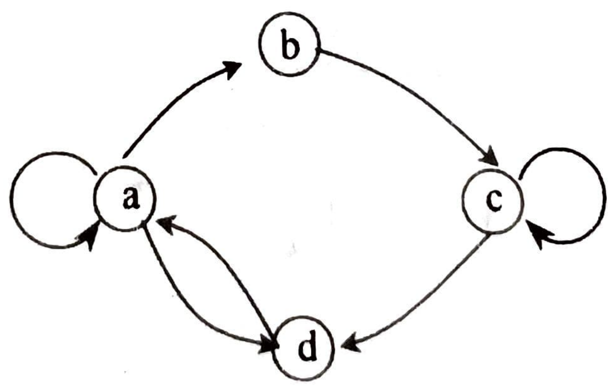


Figure 2. Digraph of the Relation

The indegrees and outdegrees of all vertices are given in the following table.

	a	b	c	d
Indegree	2	1	2	2
Outdegree	3	1	2	1

Exercises

1. Given A = { 1, 2, 3, 4 } and B = { x, y, z }. Consider the following relation from A to B.

   R = { (1, y), (1, z), (3, y), (4, x), (4, z) }.

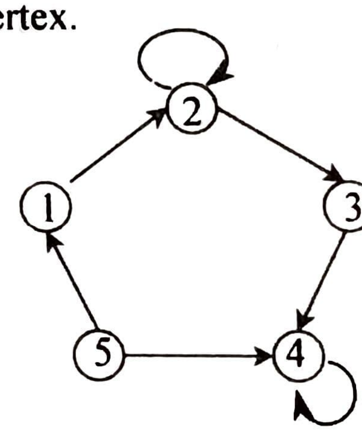
   Determine the matrix of the relation.

   (Nov '98, B.E., Bharathiar Dasan Uni.)

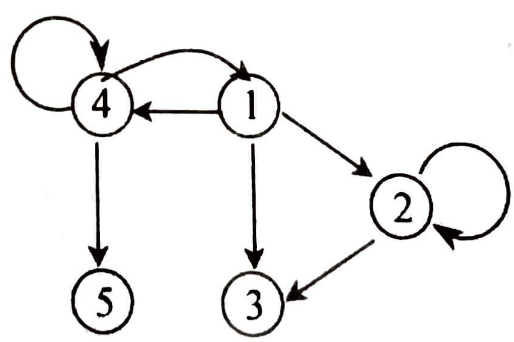
2. Let A = { 1, 2, 3, 4 }. Find the relation R on A determined by the matrix.

$$M_R = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

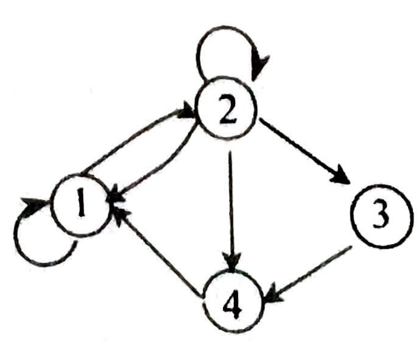
3. Find the relation determined by the following digraph. Find the indegree and outdegree for each vertex.



4. Find the relation determined by the following digraph.



5. Find the relation determined by the following digraph.



6. If A = { 1, 2, 3, 4 } and R = { (1, 1), (1, 3), (2, 3), (3, 2), (3, 3), (4, 3) }, draw the digraph of R.

Answers

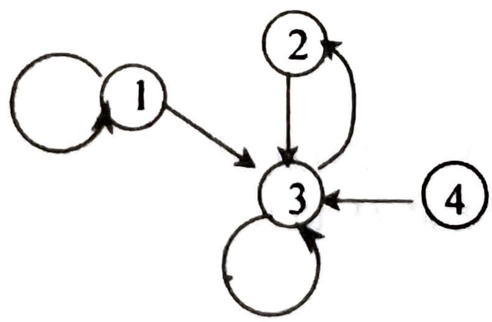
1.

$$M_R = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

2. R = { (1, 1), (1, 3), (2, 3), (3, 1), (4, 1), (4, 2), (4, 4) }.
3. A = { 1, 2, 3, 4, 5 } and
   R = { (1, 2), (2, 2), (2, 3), (3, 4), (4, 4), (5, 1), (5, 4) }

	1	2	3	4	5
Indegree	1	2	1	3	0
Outdegree	1	2	1	1	2

4. A = { 1, 2, 3, 4, 5 } and
   R = { (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (4, 1), (4, 4), (4, 5) }
5. A = { 1, 2, 3, 4 } and
   R = { (1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4), (4, 1) }.
6.



§ 4. OPERATIONS ON RELATIONS

As relations from A to B are subsets of A × B, the usual operations, on sets such as complementation, intersection and union, can be applied to relations also.

Definitions

Let R and S be relations from A to B. Then

i) R^{2} = { (a, b) ∈ A × B | (a, b) ∉ R } is the complement of the relation R.

ii) R ∩ S = { (a, b) ∈ A × B | (a, b) ∈ R and (a, b) ∈ S } is the intersection of the relations R and S.

iii) $R \cup S = \{ (a, b) \in A \times B \mid (a, b) \in R \text{ or } (a, b) \in S \}$ is the union of the relations R and S.

Given relations R and S the above definitions associate new relations from A to B. Given a relation R from A to B, we can get a relation from B to A by reversing the order of elements in each pair $(a, b) \in R$.

Definition

Every relation R from a set A into a set B has an inverse relation R^{-1} from B to A which is defined by

$$R^{-1} = \{ (b, a) \mid (a, b) \in R \}$$

i.e., the inverse relation R^{-1} consists of those ordered pairs, which when reversed belong to R.

Note

1. The relation R from a set A into a set B is a subset of the Cartesian product $A \times B$. The inverse relation R^{-1} is a subset of the Cartesian product $B \times A$.
2. Clearly $(R^{-1})^{-1} = R$.
3. If R is the relation 'husband of' then R^{-1} is the relation 'wife of' in the set of all human beings.

Illustrations

1. Let $A = \{ a, b, c\}$ and $B = \{1, 2 \}$

   Then $R = \{ (a, 1), (a, 2), (c, 1) \}$ is a relation from A to B. The inverse relation of R is $R^{-1} = \{ (1, a), (2, a), (1, c) \}$.

2. Let $A = \{ a, b, c \}$

   Then $R = \{ (a, a), (a, b), (a, c), (b, c) \}$ is a relation in A. The inverse relation of R is $R^{-1} = \{ (a, a), (b, a), (c, a), (c, b) \}$

The relation between the domain and range of a relation R and the domain and range of R^{-1}.

R^{-1} consists of the same pairs as are in R but in the reverse order. So the first element in an element of R will be a second element in an element of R^{-1} Hence the domain of R is the range of R^{-1}. Similarly we can prove that the range of R is the domain of R^{-1}.

Suppose that A, B and C are sets, R is a relation from A to B and S is a relation from B to C. We can define a new relation, the composition of R and S written as S o R. The relation S o R is a relation from A to C and is defined as follows : If x is in A and z is in C, then x (S o R) z if and only if for some y in B, we have x R y and y R z.

Suppose R is a relation on a set A i.e., R is a relation from a set A to itself. Then R o R is the composition of R with itself and it is written as R^2 and R^n is defined recursively by $R^n = R^{n-1} \text{ o } R$.

Example. Suppose A = { 1, 2, 3 }, B = { 2, 3, 6, 8, 12 }, C = { 13, 17, 22 } and R = { (1, 2), (1, 3), (1, 12), (2, 3), (2, 6), (2, 8), (2, 12) }

S = { (2, 13), (2, 17), (3, 13), (3, 22), (8, 22) } . Find S o R.

Solution

From the definition, we must take each element x in A with each element z in C and find whether there is an element y in B such that $(x, y) \in R$ and $(y, z) \in S$.

For example, $(1, 2) \in R$ and $(2, 13) \in S$. Then $(1, 13) \in$ S o R.

Similarly,

$(1, 2) \in R$ and $(2, 17) \in S$. Hence $(1, 17) \in$ S o R.

$(1, 3) \in R$ and $(3, 13) \in S$. Hence $(1, 13) \in$ S o R.

$(1, 3) \in R$ and $(3, 22) \in S$. Hence $(1, 22) \in$ S o R.

$(2, 3) \in R$ and $(3, 13) \in S$. Hence $(2, 13) \in$ S o R.

$(2, 3) \in R$ and $(3, 22) \in S$. Hence $(2, 22) \in$ S o R.

$(2, 8) \in R$ and $(8, 22) \in S$. Hence $(2, 22) \in$ S o R.

Hence S o R = { (1, 13), (1, 17), (1, 22), (2, 13), (2, 22) }.

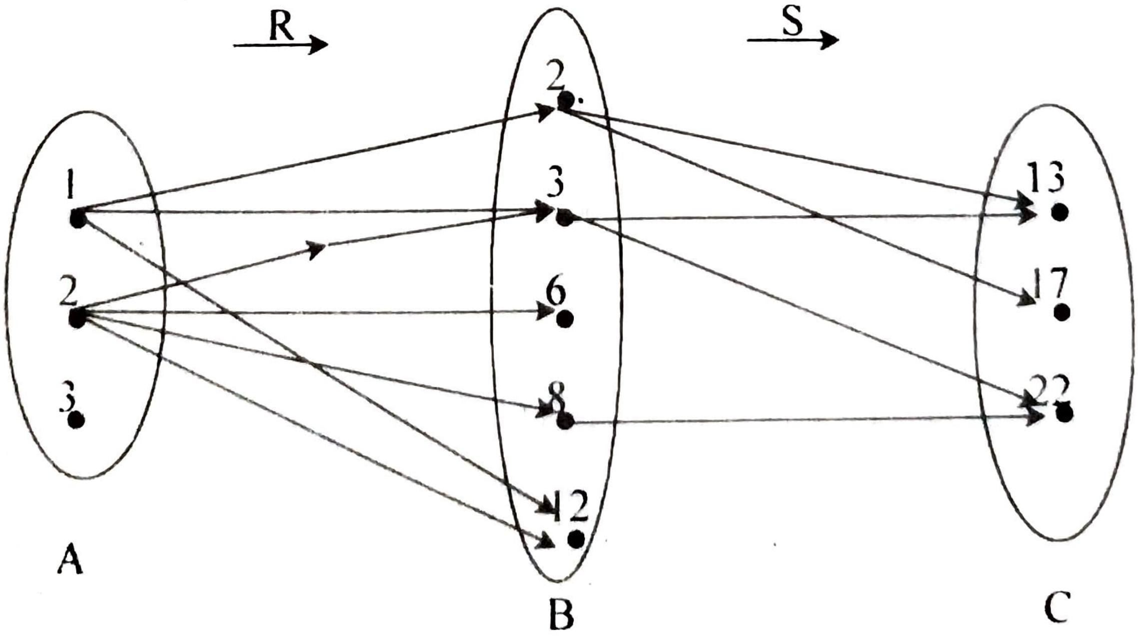


Figure 3. Composition Of Relations. S o R.

The above result can be easily obtained from a figure. (see Figure 3.)

Theorem 1. Associative Law for Composition of Relations

Let A, B, C and D be sets. Suppose R is a relation from A to B, S is a relation from B to C and T is a relation from C to D. Then

$$(T \circ S) \circ R = T \circ (S \circ R).$$

Proof Let (a, d) be an ordered pair in $(T \circ S) \circ R$.

This means that there is an element b in B such that

$$(a, b) \in R \text{ and } (b, d) \in T \circ S.$$

Since $(b, d) \in T \circ S$, there exists an element c in C such that

$$(b, c) \in S \text{ and } (c, d) \in T.$$

As $(a, b) \in R$ and $(b, c) \in S$, $(a, c) \in S \circ R$.

$(a, c) \in S \circ R$ and $(c, d) \in T \Rightarrow (a, d) \in T \circ (S \circ R)$

Hence $(T \circ S) \circ R \subset T \circ (S \circ R)$ ....(1)

Similarly we can show that

$$T \circ (S \circ R) \subset (T \circ S) \circ R$$ ....(2)

From the inclusion relations (1) and (2), we have that

$$(T \circ S) \circ R = T \circ (S \circ R)$$

Theorem 2

Let A, B and C be sets. R is a relation from A to B and S is a relation from B to C. *Then* $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

Proof Let $(c, a) \in (S \circ R)^{-1}$. Then $(a, c) \in S \circ R$.

This means that there is an element b in B such that $(a, b) \in R$ and $(b, c) \in S$.

$\therefore (b, a) \in R^{-1}$ and $(c, b) \in S^{-1}$.

As $(c, b) \in S^{-1}$ and $(b, a) \in R^{-1}$, $(c, a) \in R^{-1} \circ S^{-1}$

$\therefore (S \circ R)^{-1} \subset R^{-1} \circ S^{-1}$ ....(1)

Similarly $R^{-1} \circ S^{-1} \subset (S \circ R)^{-1}$ ....(2)

From the inclusion relations (1) and (2), we have

$$(S \circ R)^{-1} = R^{-1} \circ S^{-1}$$

Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be two finite sets.

Let R be a relation from A to B. i.e., $R \subset A \times B$. Then we have seen that R can be represented by the matrix $M_R = (m_{ij})_{m \times n}$

$$\text{where } m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

We obtain the matrices of inverse relation of R and composition of two relations as follows :

1. If R^{-1} is the inverse relation of R, then $M_{R^{-1}} = (M_R)^T$, where $(M_R)^T$ is the transpose of the matrix M_R.
2. If R is a relation from A to B and S is a relation from B to C, where $A = \{ a_1, a_2, \ldots, a_n \}$, $B = \{ b_1, b_2, \ldots, b_p \}$, $C = \{ c_1, c_2, \ldots, c_m \}$ are finite sets, let M_R and M_S be the matrices representing R and S respectively. We note that $(a_i, c_j) \in S \circ R$ if and only there is at least one $b_k \in B$ such that $(a_i, b_k) \in R$ and $(b_k, c_j) \in S$. Thus the $(i, j)^{th}$ entry in M_{SoR} is one if there is one k such that $(i,k)^{th}$ entry in M_R and $(k, j)^{th}$ entry in M_S are 1. When we scan the i^{th} row of M_R and j^{th} column of M_S, suppose we come across at least one k such that $(i, k)^{th}$ entry of M_R = $(k, j)^{th}$ entry of $M_S = 1$. Then $(i,j)^{th}$ entry of M_{SoR} is one. Otherwise it is zero. If $M_R = (a_{ij})_{m \times p}$, $M_S = (b_{ij})_{p \times n}$ and $M_{SoR} = (z_{ij})_{m \times n}$, then $z_{ij} = \max \{ a_{ik} . b_{kj} : k = 1, 2, \ldots p \}$ for all $1 \leq i \leq m$; $1 \leq j \leq n$. The matrix obtained from M_R and M_S, by this way, is denoted by $M_R \odot M_S$. Thus $M_{SoR} = M_R \odot M_S$.

For example if $M_R = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$ and $M_S = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

then $M_R \odot M_S = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

Note that $(1, 1)^{th}$ entry of $M_R \odot M_S = \max \{ a_{1k} \, b_{k1} : k = 1, 2, 3, 4, 5\}$
$= \max \{ 0, 0, 0, 1, 0 \} = 1$

$(1, 2)^{th}$ entry of $M_R \odot M_S = \max \{ a_{1k}, b_{k2} : k = 1, 2, 3, 4, 5 \}$
$= \max \{ 0, 0, 1, 0, 0 \} = 1.$

Here we introduce another binary operation on Boolean Matrices. If $M_1 = [a_{ij}]$; $M_2 = [b_{il}]$, where $a_{ij}, b_{ij} \in \{0, 1\}$, we define a matrix

$$M_1 \vee M_2 = [c_{ij}], \text{ where } c_{ij} = \max \{ a_{ij}, b_{ij} \}.$$

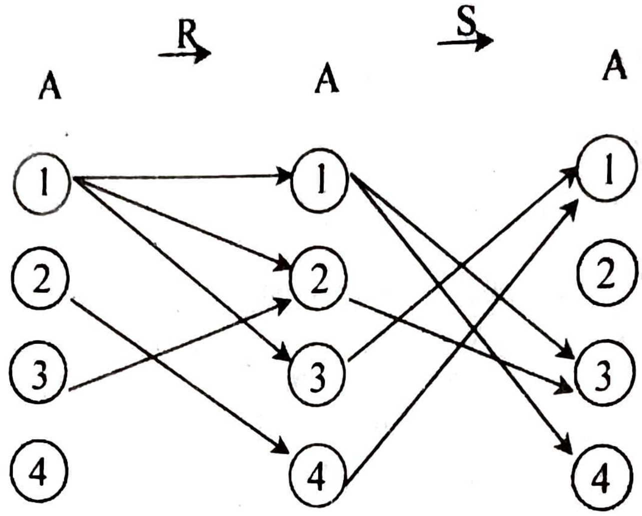
For example,

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \vee \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Worked Examples

W.E.1. Given A = { 1, 2, 3, 4 } *and* R = { (1,2), (1, 1); (1, 3), (2, 4), (3, 2) } *and* S = { (1, 4), (1, 3), (2, 3), (3, 1), (4, 1) } *are relations on* A, *find* S o R.

Solution



S o R = { (1, 3), (1, 4), (1, 1), (2, 1), (3, 3) }.

Matrix Method

Matrix $M_R = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $M_S = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

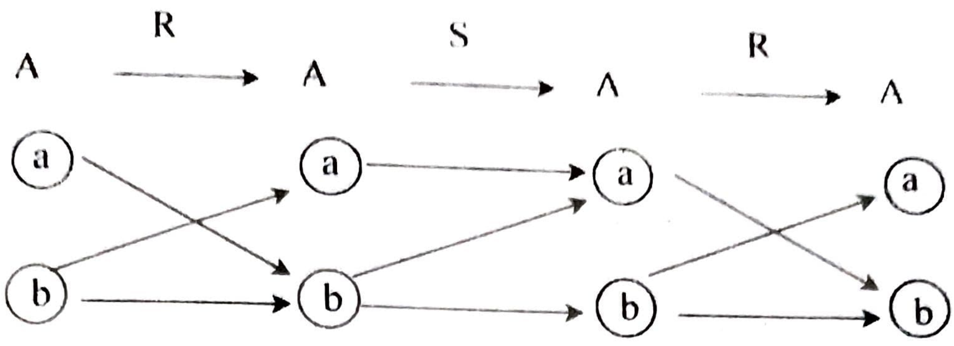
Then

$$M_{SoR} = M_R \odot M_S = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

S o R = { (1, 1), (1, 3), (1, 4), (2, 1), (3, 3) }.

W.E.2. Let A = { a, b }, let R = { (a, b), (b, a), (b, b) } and S = { (a, a), (b, a), (b, b) } be relations on A. Find S o R and R o S. Comment on your result.

Solution



S o R = { (a, a), (a, b), (b, a), (b, b) }
R o S = { (a, b), (b, a), (b, b) }

We observe that S o R ≠ R o S.

Matrix Method

$$M_R = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad M_S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

So $$M_{SoR} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad M_{RoS} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

So S o R = { (a, a), (a, b), (b, a), (b, b) }, while
R o S = { (a, b), (b, a), (b, b) }

W.E.3. Let A = { 1, 2, 3, 4 }.
Let R = { (1, 1), (1, 2), (2, 3), (2, 4), (3, 4), (4, 1), (4, 2) } and S = { (3, 1), (4, 4), (2, 3), (2, 4), (1, 1), (1, 4) } be two relations on A.

a) Is (1, 3) ∈ R o R ?
b) Is (4, 3) ∈ S o R ?
c) Is (1, 1) ∈ R o S ?
d) Compute S o R, R o S, R o R, and S o S

Solution

a) As (1, 2) ∈ R and (2, 3) ∈ R, we have (1, 3) ∈ R o R.
b) As (4, 2) ∈ R and (2, 3) ∈ S, we have (4, 3) ∈ S o R.
c) As (1, 1) ∈ S and (1, 1) ∈ R, we have (1, 1) ∈ R o S.
d) S o R = { (1, 1), (1, 4), (1, 3), (2, 1), (2, 4), (3, 4), (4, 1), (4, 3), (4, 4) }
R o S = { (1, 1), (1, 2); (2, 4), (2, 1), (2, 2), (3, 1), (3, 2), (4, 1), (4, 2) }
R o R = { (1, 1), (1, 2), (1, 3), (1, 4), (2, 4), (2, 1), (2, 2), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (4, 4) }

$S \circ S = \{ (3, 1), (3, 4), (4, 4), (2, 1), (2, 4), (1, 1), (1, 4) \}$.

(Verify these results by matrix method)

Exercises

1. Find the inverse of the relations given in problems 1, 2 and 4 of exercises of section § 2.
2. If R and S are relations from A to B, show that
   i) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$.
   ii) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$.
   iii) If $R \subseteq S$, then $R^{-1} \subseteq S^{-1}$
3. Let A = { 1, 2, 3, 4 } and R = { (1, 1), (1, 2), (2, 3), (3, 4) } and S = { (2, 3), (2, 1), (4, 3) } be relations on A. Find R o S and S o R.
4. Two relations R and S on { a, b, c } are given by their matrices M_R and M_S. Find R o S.

$$M_R = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad M_S = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

5. If R is given by the matrix $M_R = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ find the matrices of R o R and R o R o R.

Answers

1. a) $R^{-1} = \{ (2, 3), (2, 5), (4, 5), (2, 7), (4, 7), (6, 7), (2, 9), (4, 9), (6, 9), (8, 9) \}$
   b) $R^{-1} = \{ (6, 2), (10, 2), (3, 3), (6, 3), (10, 5) \}$
   c) $R^{-1} = \{ (3, 3), (2, 6), (1, 9) \}$

3. R o S = {(1, 1), (1, 3), (3, 3)}
   S o R = { (2, 4), (2, 1), (2, 2), (4, 4)}

4. $M_{RoS} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

   R o S = {(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, b), (c, c) }.

1. $M_{RoR} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

§ 5. EQUIVALENCE RELATIONS

In many of the applications to computer science and applied mathematics, we are more concerned with relations on a set A then relations from a set A to a different set B. In this section, we study relations on a set A, especially we study certain special types of relations on A.

First, we note that if A is a set, then $\{ (a, a) \mid a \in A \}$ is also a relation, we denote this relation by Δ, and it is called the identity relation on A.

$$\Delta = \{(a, a) : a \in A\}.$$

Definitions

A relation h on a set A is said to be

i) *reflexive*	if	$(a, a) \in R$, for all $a \in A$.
ii) *irreflexive*	if	$(a, a) \notin R$, for all $a \in A$.
iii) *symmetric*	if	$(a, b) \in R \Rightarrow (b, a) \in R$.
iv) *antisymmetric*	if	$a \neq b$ and $(a, b) \in R \Rightarrow (b, a) \notin R$.
v) *transitive*	if	$(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$.
vi) *assymetric*	if	$(a, b) \in R \Rightarrow (b, a) \notin R$.

Note A relation R on the set A is

i) reflexive	if and only if	$\Delta \subset R$.
ii) irreflexive	if and only if	$R \cap \Delta = \phi$.
iii) symmetric	if and only if	$R^{-1} = R$.
iv) antisymmetric	if and only if	$R \cap R^{-1} \subset \Delta$.
v) transitive	if and only if	$R \circ R \subseteq R$.

Definition

A relation R on a set A is said to be an *equivalence relation* if it is reflective, symmetric and transitive.

Definition

A relation R on a set A is said to be a *partial order relation* if it is reflexive, antisymmetric and transitive.

Remark. We study more about the partial order relations in the Chapter X.

Note

1. If $(a, a) \notin R$ for at least $a \in A$, then the relation R is not reflexive.
2. If $(a, a) \in R$ for at least one $a \in A$, then R is not irreflexive.
3. If $(a, b) \in R$, but $(b, a) \notin R$, for at least one past $(a, b) \in A \times A$, then R is not symmetric.

Illustrations

1. Let A be the set of all triangles in the Euclidean plane. The relation R on A is defined as follows : for $x, y \in A$, $x R y$ if and only if "x is congruent to y". This relation is reflexive, symmetric and transitive. So it is an equivalence relation.

2. Let A be the set of all straight lines in the Euclidean plane. If relation R and S on A are defined by,

   $x R y$ if and only if 'x is parallel to y'.

   $x S y$ if and only if 'x is perpendicular to y'

   then R is reflexive, symmetric and transitive (so it is an equivalence relation), while the relation S is not reflexive (infact, it is irreflexive as no straight line is perpendicular to itself.), and not transitive, but it is symmetric.

3. Let $A = \{1, 2, 3, 4\}$ and

   $R = \{(1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (2, 2), (3, 3)\}$

   $S = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

   Then R is symmetric, but not reflexive (as $(4, 4) \notin R$) and also not transitive, as $(3, 1), (1, 2) \in R$, but $(3, 2) \notin R$. The relation S is symmetric, transitive, but not reflexive as $(4, 4) \notin S$.

4. Let N be the set of all natural numbers. Define a relation R on N as follows : a R b if and only if a divides b (i.e., $b = ka$ for some positive integer k). Then R is reflexive, transitive, but not symmetric, as $(4, 8) \in R$, but $(8, 4) \notin R$. Note that $(a, b) \in R$ and $(b, a) \in R$, then $b = na$ and $a = mb$ for some positive integers m, n. So $a = mb = mna$. As a, b, m and n are positive integers, $a = mn\,a$ implies that $m = n = 1$. So $a = b$, then $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ i.e, R is antisymmetric.

5. If $A = \{1, 2, 3, 4\}$, then

   i) the relation $\{(1, 2), (2, 4)\}$ is not reflexive, not symmetric, and not transitive.

ii) The relation {(1, 1), (2, 2), (3, 3), (4, 4), (1, 3), (3, 2)} is reflexive, but neither symmetric nor transitive.

iii) The relation {(1, 1), (1, 3), (3, 1), (3, 4), (4, 3)} is symmetric, but neither reflexive nor transitive.

iv) The relation {(1, 1), (1, 3)} is transitive, but neither reflexive nor symmetric.

v) The relation {(1, 1), (2, 2), (3, 3), (4, 4), (1, 3), (3, 1), (3, 4), (4, 3)} is reflexive, symmetric but not transitive.

vi) The relation {(1, 1), (2, 2), (3, 3), (4, 4), (1, 3)} is reflexive, transitive but not symmetric.

vii) The relation {(1, 1), (2, 2), (2, 3), (3, 2), (3, 3)} is symmetric, transitive but not reflexive.

viii) The relation {(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1)} is an equivalence relation.

6. Let A be a non empty set. Let $\Delta = \{(a, a) : a \in A\}$. Then Δ is reflexive, symmetric (as $\Delta^{-1} = \Delta$) and transition (as $\Delta \circ \Delta = \Delta$). So the identity relation Δ is an equivalence relation.

Theorem 3

Let R and S be relations on A. Then

1) if R is reflexive, then $R \cup S$ is reflexive
2) if R and S are reflexive, then $R \cap S$ is reflexive.
3) if R and S are symmetric, then $R \cap S$ and $R \cup S$ are symmetric.
4) if R and S are transitive, then $R \cap S$ is transitive.
5) if R and S are equivalence relations, then so is $R \cap S$.

Proof

1. if R is reflexive, then $\Delta = \{(a, a) \mid a \in A\} \subseteq R$. So $\Delta \subseteq R \cup S$. Then $R \cup S$ is reflexive.

2. If both R and S reflexive, then $\Delta \subseteq R$ and $\Delta \subseteq S$ and hence $\Delta \subseteq R \cap S$. Thus $R \cap S$ is reflexive.

3. Assume that both R and S to be symmetric. Then if $(a, b) \in R \cap S$, then $(a, b) \in R$ and $(a, b) \in S$.
As $(a, b) \in R$ and as R is symmetric, $(b, a) \in R$.
As $(a, b) \in S$ and as S is symmetric, $(b, a) \in S$.
Then $(a, b) \in R \cap S \Rightarrow (b, a) \in R \cap S$. So $(R \cap S)$ is also symmetric.
Similarly, we can show that $R \cup S$ is also symmetric.

4. Assume that R and S are transitive. Let $(a, b), (b, c) \in R \cap S$. As $(a, b), (b, c) \in R$, we have $(a, c) \in R$. As $(a, b), (b, c) \in S$, we have $(a, c) \in S$. Thus $(a, b), (b, c) \in R \cap S \Rightarrow (a, c) \in R \cap S$ and $R \cap S$ is also transitive.

5. If R and S are equivalence relations on A, then by (2), (3) and (4), it follows that $R \cap S$ is also an equivalence relation.

Note

1. R and S are transitive need not imply $R \cup S$ is transitive. For example let $A = \{1, 2, 3, 4\}$, $R = \{(1, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$ and $S = \{(2, 3), (3, 3), (3, 4), (4, 3), (4, 4)\}$. Then $(1, 2), (2, 3) \in R \cup S$, but $(1, 3) \notin R \cup S$.

2. If R and S are relations, then $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$ and $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$ (see the problem 3 in Exercises in section § 4). If R and S are symmetric then $R^{-1} = R$ and $S^{-1} = S$. Hence $(R \cap S)^{-1} = R^{-1} \cap S^{-1} = R \cap S$. So $R \cap S$ is also symmetric. In the same way $R \cup S$ is also symmetric.

Theorem 4

Let R be a relation on a set A. Then

1. if R is reflexive, then R^{-1} is also reflexive.
2. if R is transitive, then R^{-1} is also transitive.
3. if R is an equivalence relation, then R^{-1} is also an equivalence relation.

Proof

1. R reflexive $\Rightarrow \Delta \subseteq R \Rightarrow \Delta^{-1} \subseteq R^{-1} \Rightarrow \Delta \subseteq R^{-1}$ as $\Delta^{-1} = \Delta$
$\Rightarrow R^{-1}$ is reflexive.

2. R is transitive $\Rightarrow R \circ R \subseteq R \Rightarrow (R \circ R)^{-1} \subseteq R^{-1}$
$\Rightarrow R^{-1} \circ R^{-1} \subseteq R^{-1}$, as $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$.
$\Rightarrow R^{-1}$ is transitive.

3. If R is an equivalence relation, then it is symmetric. So $R = R^{-1}$. As $R^{-1} = R$ and R is an equivalence relation, R^{-1} $(= R)$ is an equivalence relation.

Note. We can also prove 2 (Theorem 4) as follows :

If R is transitive, then $(a, b), (b, c) \in R^{-1}$ $\Rightarrow (b, a), (c, b) \in R$

$\Rightarrow (c, a) \in R$, as R is transitive.

$\Rightarrow (a, c) \in R^{-1}$

$\Rightarrow R^{-1}$ is transitive.)

Let m be a given positive integer, and N be the set of all natural numbers. Let $x, y \in N$. We say that $x - y$ is divisible by m if $x - y = km$ for some integer k (k need not be a positive integer). If $x - y$ is divisible by m, then we write $x \equiv y \pmod{m}$ and say that x is congruent to y modulo m. Also $x \equiv y \pmod{m}$ if and only if x and y leave the same remainder r when divided by m. For example we have $7 \equiv 5 \pmod{2}$, $17 \equiv 42 \bmod (15)$.

Worked Examples

W.E.1. Prove that the relation "congruence modulo m" over the set of positive integers is an equivalence relation. (Apr '97, B.E., M.U.).

Solution

Let N be the set of all positive integers and m be a given positive integer. We define the 'congruence modulo m' relation on N as follows :

For $x, y \in N$, $x \equiv y \pmod{m}$ if and only if $x - y$ is divisible by m.

(i.e., $x - y = km$ for some integer $k \in Z$).

Let $x, y, z \in N$. Then

i) as $x - x = 0 = 0.m$. So $x \equiv x \pmod{m}$, for all $x \in N$. Thus this relation is reflexive.

ii) $x \equiv y \pmod{m} \Rightarrow x - y = km$ for some integer k.

$\Rightarrow y - x = (-k)m$.

$\Rightarrow y \equiv x \pmod{m}$

So the relation is also symmetric.

iii) $x \equiv y \pmod{m}$ and $y \equiv z \pmod{m}$

$\Rightarrow x - y = km$ and $y - z = lm$ for some integers k and l.

$\Rightarrow (x - y) + (y - z) = (k + l)m$

$\Rightarrow x - z = (k + l)m$

$\Rightarrow x \equiv z \pmod{m}$ as $k + l$ is also an integers.

So the relation is transitive.

As it is reflexive, symmetric and transitive, the relation 'congruence modulo m' is an equivalence relation.

W.E.2. Let R denote a relation on the set of all ordered pairs of positive integers, by

(x, y) R (u, v) *if and only if* $xv = yu$.

Show that R is an equivalence relation.

(Nov '96, Nov '97, M.C.A., M.U.; Oct '98, B.E., M.U.; Nov '97, B.E., Bharathi Dasan Uni.)

Solution :

If x, y, u, v are positive integers, it is given that (x, y) R (u, v) if and only if $xv = yu$.

(i) As $xy = yx$ is true for all positive integers x and y, we have (x, y) R (x, y), for all ordered pair (x, y) of positive integers. So the relation R is reflexive.

(ii) (x, y) R $(u, v) \Rightarrow xv = yu$
$\Rightarrow yu = xv$
$\Rightarrow uy = vx$
$\Rightarrow (u, v)$ R (x, y)

So R is symmetric.

(iii) Let x, y, u, v, m, n be positive integers.

(x, y) R (u, v) and (u, v) R (m, n)
$\Rightarrow xv = yu$ and $un = vm$
$\Rightarrow xvun = yuvm$
$\Rightarrow xn = ym$, by cancelling vu, (note that $uv \neq 0$)
$\Rightarrow (x, y)$ R (m, n)

So R is transitive.

As R is reflexive, symmetric and transitive, it is an equivalence relation.

W.E.3. Let a relation R be defined on the set of all real numbers by 'if x, y are real numbers, x R y $\Leftrightarrow$ x − y is a rational number'. Show that the relation R is an equivalence relation.

Solution

It is given that

x R y if and only if $x - y$ is a rational number.

(i) As $x - x = 0$ is a rational number for all real numbers x, the relation R is reflexive.

(ii) Let x R y. Then x − y = z for some rational number. Now y − x = −z is also a rational number. So y R x. Thus x R y ⇒ y R x and so R is symmetric.

(iii) Let x R y and y R z. Then x − y = r_1 and y − z = r_2 for some rational numbers r_1 and r_2. Now x − z = (x − y) + (y − z) = $r_1 + r_2$ is also a rational number. So x R z. Thus x R y and y R z ⇒ x R z and R is transitive.

As R is reflexive, symmetric and transitive, it is an equivalence relation.

[Note that 0 is a rational number. If r_1 and r_2 are rational numbers then $r_1 + r_2$ and $-r_1$ are also rational numbers.]

Exercises

In problem 1 to 6, let A = {1, 2, 3, 4}. Determine whether the relation R is reflexive, irreflexive, symmetric, asymmetric, antisymmetric or transitive.

1. R = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)}
2. R = {(1, 3), (1, 1), (3, 1), (1, 2), (3, 3), (4, 4)}
3. R = ϕ.
4. R = {(1, 2), (1, 3), (3, 1), (1, 1), (3, 3), (3, 2), (1, 4), (4, 2), (3, 4)}
5. R = {(1, 1), (1, 2), (1, 3), (1, 5), (2, 3), (4, 4), (4, 2), (4, 3), (5, 3)}
6. R = {(1, 2), (1, 4), (2, 1), (2, 3), (2, 4), (3, 2), (4, 1), (4, 2)}
7. Relations R_1, R_2, R_3 and R_4 are defined on N, the set of natural numbers, as follows :

   i. (x, y) ∈ R_1 ⇔ x is greater than y

   ii. (x, y) ∈ R_2 ⇔ x is a multiple of y

   iii. (x, y) ∈ R_3 ⇔ xy is a square of an integer

   iv. (x, y) ∈ R_4 ⇔ x + 3y = 12.

   State whether or not each of these relations is

   a. reflexive b. symmetric

   c. antisymmetric d. transitive.
8. Let L be the set of all straight lines in the Euclidean plane and R be the relation in L defined by

   x R y ⇔ x is perpendicular to y.

   Is R reflexive? Symmetric? Antisymmetric? Transitive?
9. Give one example of a relation R on A = {1, 2, 3, 4} which is neither symmetric nor anti symmetric.
10. Give one example of a relation R on A = {1, 2, 3, 4} which is symmetric, transitive, but not reflexive.

11. Let m be a positive integer. Prove that the relation 'congruence modulo m' on N, the set of all positive integers, given by

$x \equiv y$ if and only if $x - y$ is divisible by m is an equivalence relation.
(Apr'97, B.E., M.U.; Apr'98, M.C.A., M.U.)

12. Let X = {1, 2, ..., 7} and R = {(x, y) | x − y is divisible by 3}. Show that R is an equivalence relation. Draw the digraph of R. ('96, M.C.A., M.U.)

Answers

	Reflexive	Irreflexive	Symmetric	Asymmetric	Antisymmetric	Transitive
1.	Yes	No	Yes	No	No	Yes
2.	No	No	No	No	No	No
3.	No	Yes	Yes	Yes	Yes	Yes
4.	No	No	No	No	No	Yes
5.	No	No	No	No	Yes	Yes
6.	No	No	Yes	No	No	No

7. i. antisymmetric and transitive.
   ii. reflexive, antisymmetric and transitive
   iii. reflexive, symmetric and transitive.

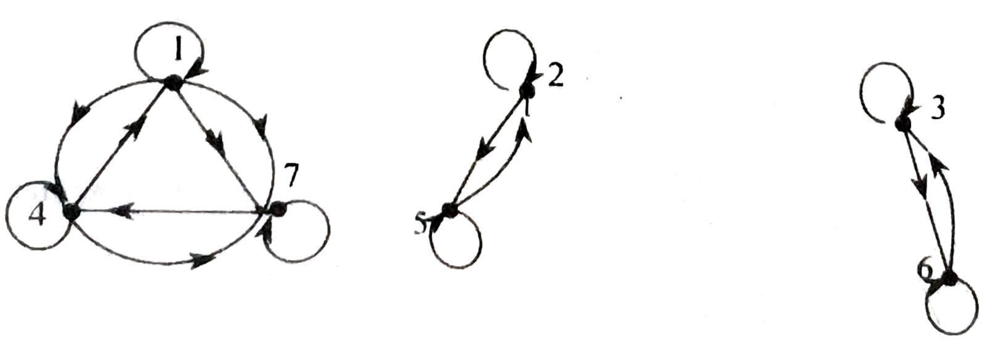
[If n is a positive integer, then n can be written in the form. $n = ab^2$, where a, b are positive integers and either a = 1 or a is not divisible by p^2, for any prime number p. The integer a is known as square free part of n.

If $n = ab^2$, $m = cd^2$ and $nm = k^2$ for some integer k, then a = c. Use this property to prove the transitive property of R]

   iv. R = {(3, 3), (6, 2), (9, 1)}. R is antisymmetric and transitive.
8. Symmetric only
9. R = {(1, 2), (2, 1), (2, 4)} is one such example.
10. One such example is

{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}

12.



§ 6. CLOSURES AND WARSHALL'S ALGORITHM

R C R1

If R is a relation on a set A, it may well happen that R may not have some of the important relational properties such as reflexivity, symmetry, transitivity. If R does not possess a particular property, we may like to find the smallest relation R_1 on A so that R_1 possesses the desired property and R_1 contains R. The smallest such relation R_1 is called the closure of R with respect to the property in question. In this section we deal with the reflexive closure, the symmetric closure, and the transitive closure of a relation R on a set A.

REFLEXIVE CLOSURE

Let R be a relation on A. We know that R is reflexive if and only if the relation $\Delta = \{(a, a) \mid a \in A\}$ is contained in R. i.e., R is reflexive$\Leftrightarrow \Delta \subset R$. So if a given relation R is not reflexive, then $R \cap \Delta \neq \Delta$. The relation $R_1 = R \cup \Delta$ is the smallest reflexive relation on A, containing R. Thus the reflexive closure of a relation R is $R \cup \Delta$.

SYMMETRIC CLOSURE

We know that a relation R is symmetric if $(x, y) \in R \Leftrightarrow (y, x) \in R$ i.e., if $R^{-1} = \{(y, x) \mid (x, y) \in R\} = R$. Let R be a given relation on A. Any symmetric relation S on A that contains R should contain R^{-1} also. So $R \cup R^{-1} \subset S$, for any symmetric relation S that contains R. As $(R \cup R^{-1})^{-1} = R \cup R^{-1}$, $R \cup R^{-1}$ itself is a symmetric relation on A, containing R. Thus the symmetric closure of R is $R \cup R^{-1}$

TRANSITIVE CLOSURE

Let R be a given relation on A. As $A \times A$ itself is a transitive relation that contains R, the smallest transitive relation R_1, that contains R exists. The relation R_1 is the intersection of all transitive relations on A that contain R.

Example. Let $A = \{1, 2, 3, 4\}$ and R_1 and R_2 be the relations on A given by $R_1 = \{(1, 1), (1, 2), (2, 3), (3, 4)\}$ and $R_2 = \{(1, 2), (2, 2), (2, 3), (3, 2), (4, 1), (4, 4)\}$ then

i. reflexive closure of $R_1 = R_1 \cup \Delta$.

$$= \{(1, 1), (1, 2), (2, 3), (3, 4)\} \cup \{(1, 1), (2, 2), (3, 3), (4, 4)\}$$
$$= \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 3), (3, 4)\}$$

ii. the symmetric closure of $R_1 = R_1 \cup R_1^{-1}$

$$= \{(1, 1), (1, 2), (2, 3), (3, 4)\} \cup \{(1, 1), (2, 1), (3, 2), (4, 3)\}$$
$$= \{(1, 1), (1, 2), (2, 3), (3, 4), (2, 1), (3, 2), (4, 3)\}$$

iii. the reflexive closure of $R_2 = R_2 \cup \Delta$

$$= \{(1, 2), (2, 2), (2, 3), (3, 2), (4, 1), (4, 4)\} \cup \{(1, 1), (2, 2), (3, 3), (4, 4)\}$$
$$= \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 3), (3, 2), (4, 1)\}$$

iv. the symmetric closure of $R_2 = R_2 \cup R_2^{-1}$

$$= \{(1, 2), (2, 2), (2, 3), (3, 2), (4, 1), (4, 4)\}$$
$$\cup \{(2, 1), (2, 2), (3, 2), (2, 3), (1, 4), (4, 4)\}$$
$$= \{(1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (4, 1), (1, 4), (4, 4)\}$$

Before explaining how to find the transitive closure of a relation R, let us have a close look on the relation R^n.

Let R be a relation on the set A. Then we can find $R^2 = R \circ R$, $R^3 = R^2 \circ R, \ldots, R^n = R^{n-1} \circ R$, etc. We recall that $(a, b) \in R^2$ if and only if there is an element $c \in A$ such that $(a, c) \in R$ and $(c, b) \in R$. In fact $(a, b) \in R^n$ if and only if we can find a sequence $x_1, x_2, \ldots, x_{n-1}$ in A such that $(a, x_1), (x_1, x_2), \ldots, (x_k, x_{k+1}), \ldots, (x_{n-1}, b)$ are all in R. The elements $x_1, x_2, \ldots, x_{n-1}$ need not be distinct. The sequence $a, x_1, x_2, \ldots, x_{n-1}, b$ is said a chain of length n from a to b, in R, if $(a, x_1) \in R, (x_1, x_2) \in R, \ldots, (x_{n-1}, b) \in R$. The elements $x_1, x_2, \ldots, x_{n-1}$ are called internal vertices of this chain. If $a, x_1, \ldots, x_{n-1}, b$ is chain from a to b in R, and if the elements $x_1, x_2, \ldots, x_{n-1}$ are all distinct (elements of A), then the chain is said to be a *path* from a to b in R.

Let $a, b \in A$. If there is a chain from a to b, in R we say that $a\ R'\ b$. Thus R' is a relation on A defined as $R' = \{(a, b) \in A \times A \mid \exists\, x_1, x_2, \ldots, x_{n-1} \in A$, for some n, such that $(a, x_1), (x_1, x_2), \ldots, (x_{n-1}, b) \in R\}$. Then clearly $R' = R \cup R^2 \cup R^3 \ldots = \bigcup_{k=1}^{\infty} R^k$.

Theorem 5

Let A be a set with $|A| = n$ and R be a relation on A. Then

$$R^{\infty} = R \cup R^2 \cup \dots \cup R^n.$$

Proof Let $(a, b) \in R^{\infty}$. Then there is a chain from a to b in R. Let $a, x_1, x_2, \dots, x_{k-1}, b$ be a shortest chain from a to b in R, (i.e., for any other chain $a, y_1, y_2, \dots, y_{m-1}, b$ from a to b in R, we have $m \geq k-1$). We claim that the internal vertices $x_1, x_2, \dots, x_{k-1}$ are all distinct. If $x_i = x_j$ for some $1 \leq i \leq j \leq k-1$, then as $(x_i, x_{j+1}) = (x_j, x_{j+1}) \in R$, we observe that $a, x_1, \dots, x_i, x_{j+1}, x_{j+2}, \dots, b$ is also a chain from a to b in R, (see Figure 4). But the length of this new chain is smaller than that of the shortest chain from a to b, which is a contradiction. Thus the internal vertices $x_1, x_2, \dots, x_{k-1}$ are all distinct.

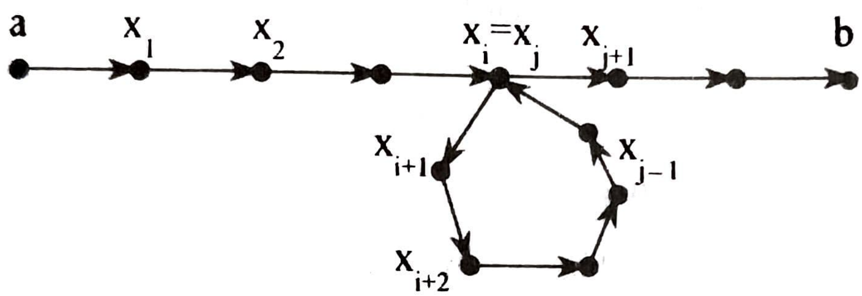


Figure 4 : A chain from a *to* b. *(Here* u●⟶●v *means* $(u, v) \in R$*).*

We also observe that $a \neq x_i$, for all $i = 1, 2, \dots, k-1$. For if $a = x_i$ for some i, then $a, x_{i+1}, \dots, x_{k-1}, b$ is shorter than shortest chain from a to b, which is again a contradiction. Thus $a, x_1, x_2, \dots, x_{n-1}$ are all distinct elements of A. As $|A| = n$, we have $k \leq n$. So the shortest chain $a, x_1, x_2, \dots, x_{k-1}$ from a to b in R is of length $\leq n$, and hence $(a, b) \in R^k$ for some $k \leq n$. Hence

$$R^{\infty} \subset R \cup R^2 \cup \dots \cup R^n.$$

As $R^{\infty} = \bigcup_{k=1}^{\infty} R^k$, it follows that $R^{\infty} = R \cup R^2 \cup \dots \cup R^n$.

Remarks

1. We have proved in the above theorem that if $(a, b) \in R^{\infty}$, then there is a path $a, x_1, \dots, x_{k-1}, b$ from a to b in R, for some $k \leq n$.

2. Let A be a set with $|A| = n$ and R be a relation on A. It may happen that if $1 \le m < n$, $R^m \neq \phi$, and $R^m \cap R^k = \phi$, $\forall k \neq m \in \{1, 2, 3, \ldots, n\}$. For example Let $A = \{1, 2, \ldots, n\}$ and $R = \{(1, 2), (2, 3), (3, 4), \ldots, (n-1, n), (n, 1)\}$ be a relation on A. Then $R^m = \{(1, m+1), (2, m+2), \ldots, (n, m)\}$ for all $m = 1, 2, 3, \ldots, n$, (addition $m + r$ is taken modulo n). Hence it is not possible to obtain a stronger result than the result given in Theorem 5.

Now we have a theorem on the transitive closure of a relation R.

Theorem 6

Let R be a relation on a set A. Then R^∞ is the transitive closure of R.

Proof $R \subset R^\infty$ follows from the definition of R^∞.

Now we claim that the relation R^∞ is transitive.

Let $(a, b), (b, c) \in R^\infty$.

Then we have a chain $a, x_1, x_2, \ldots, x_n, b$ from a to b in R and a chain $b, y_1, y_2, \ldots, y_m, c$ for b to c in R. Then $a, x_1, x_2, \ldots, x_n, b, y_1, y_2, \ldots, y_m, c$ is a chain from a to b in R. So $(a, c) \in R^\infty$. Thus $(a, b), (b, c) \in R^\infty \Rightarrow (a, c) \in R^\infty$. i.e., the relation R^∞ is transitive.

Now to show that R^∞ is the smallest transitive relation that contains R, consider a transitive relation S on A, such that $R \subset S$. Since S is transitive, we have $S^\prime \subset S$, in otherwords. $S^n \subset S$ for all positive integers n.

As $R \subset S$, $R^n \subset S^n$ for all positive integers n. Thus $R^n \subset S$ for all $n = 1, 2, \ldots$ and so $\bigcup_{k=1}^{\prime} R^n \subset S$. i.e., $R^\infty \subset S$. Hence R^∞ is the smallest transitive relation that contains R.

Remark. If A is a finite set and $|A| = n$, and R is a relation on A, then by theorems 5 and 6,

$R^\prime$ = the transitive closure of $R = R \cup R^2 \cup \ldots \cup R^n$.

Note that (i) $M_{R^2} = M_R \odot M_R$, $M_{R^3} = M_R \odot M_R \odot M_R$ etc.

(ii) i, j^{th} element of $M_R \odot M_R$ is equal to 1 if and only if row i of M_R and column j of M_R have a 1 in the same relative position k, for some k.

(iii) i, j^{th} element of $M_R \vee M_S = \max\{(i, j)^{th}$ element of M_R, $(i, j)^{th}$ element of $m_j\}$

So if M_R is the matrix of R, then the matrix $M_{R'}$ is given by the relation $M_{R'} = M_R \vee M_{R^2} \vee \ldots \vee M_{R^n}$.

Worked Examples

W.E.1. Let A = {1, 2, 3, 4} and R = {(1, 1), (1, 3), (2, 3), (3, 4), (4, 1), (4, 2)}. *Find the transitive closure of* R.

Solution

$$M_R = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$$M_{R^2} = M_R \odot M_R = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \odot \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$M_{R^3} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \qquad M_{R^4} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$M_{R'} = M_R \vee M_{R^2} \vee M_{R^3} \vee M_{R^4} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

So $R' = A \times A$.

W.E.2. Let A = {1, 2, 3, 4} *and* R *be a relation given by its matrix*

$$M_R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Find the transitive closure of R.

Solution

As | A | = 4, the transitive closure $R' = R \cup R^2 \cup R^3 \cup R^4$.

Now

$$M_R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_{R^2} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \odot \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = M_R.$$

As $M_{R^2} = M_R$, we have $M_{R^3} = M_{R^2} \odot M_R = M_R \odot M_R = M_R$.

$M_{R^4} = M_{R^3} \odot M_R = M_R \odot M_R = M_R$

Hence $M_{R'} = M_R \vee M_{R^2} \vee M_{R^3} \vee M_{R^4} = M_R \vee M_R \vee M_R \vee M_R = M_R$.

So $R'' = R$.

Remark. If $R' = R$ for a relation R, then R is a transitive relation. In this case $R^2 = R$ is transitive.

W.E.3. Let A = {1, 2, 3, 4} *and* R = {(1, 2), (2, 3), (3, 3), (3, 4), (4, 2)}. *Find the transitive closure of* R.

Solution

$$M_R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$M_{R^2} = M_R \odot M_R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$M_{R^3} = M_{R^2} \odot M_R = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$M_{R^4} = M_{R^3} \odot M_R = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$M_{R^\infty} = M_R \vee M_{R^2} \vee M_{R^3} \vee M_{R^4}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \vee \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \vee \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \vee \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

So R^∞ = {(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)}

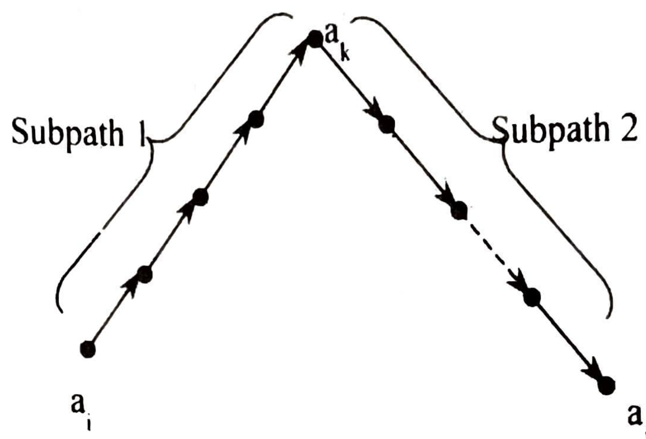
WARSHALL'S ALGORITHM

Let us now describe Warshall's algorithm, which is a more efficient algorithm for computing transitive closure of a relation.

Let $A = \{a_1, a_2, \ldots, a_n\}$ and R be a relation on A. Now for each k, $1 \le k \le n$, we define a matrix W_k as follows : The matrix W_k has a 1 in (i, j)th position if and only if there is a chain from a_i to a_j in R whose interior vertices (if any) come from the set $\{a_1, a_2, \ldots, a_k\}$. (i.e., either $(a_i, a_j) \in R$, or there is a chain $a_i, x_1, x_2, \ldots, x_m, a_j$ from a_i to a_j in R such that $x_1, x_2, \ldots, x_m \in \{a_1, a_2, \ldots, a_k\}$. We note that $W_n = M_{R^\infty}$. We define W_0 to be M_R, we compute W_k from W_{k-1} as follows :

Let $W_k = [t_{ij}]$ and $W_{k-1} = [s_{ij}]$. If $t_{ij} = 1$, then there must be a chain from a_i to a_j in R, whose interior vertices come from the set $\{a_1, a_2, \ldots, a_k\}$. So we can obtain a path, (whose interior vertices are all distinct), from a_i to a_j in R, whose interior vertices are from $\{a_1, a_2, \ldots, a_k\}$. If a_k is not an interior vertex of this path, then all the interior vertices of the path must actually come from the set $\{a_1, a_2, \ldots, a_{k-1}\}$. So $s_{ij} = 1$. If a_k is an interior vertex of this path, then a_k appears only once in this path as an interior vertex and all other interior vertices come from $\{a_1, a_2, \ldots, a_{k-1}\}$.

If the path is $a_i, x_1, x_2, \ldots, x_{l-1}, x_l, x_{l+1}, \ldots, x_m, a_j$ and $x_l = a_k$, then the sub paths $a_i, x_1, x_2, \ldots, x_{l-1}, x_l = a_k$ and $a_k = x_l, x_{l+1}, \ldots, x_m, a_j$ are paths from a_i to a_k and a_k to a_j respectively. Also the interior vertices of these subpaths are from $\{a_1, a_2, \ldots, a_{k-1}\}$.



So $s_{ik} = 1$ and $s_{kj} = 1$.

Then $t_{ij} = 1$ if and only if either $s_{ij} = 1$ or $s_{ik} = 1$ and $s_{kj} = 1$. Thus we have the following procedure for computing W_k from W_{k-1}.

Step 1 : First transfer to W_k all 1's in W_{k-1}.

Step 2 : List the locations $p_1, p_2, \ldots,$ in column k of W_{k-1} where the entry is 1 and the locations $q_1, q_2, \ldots$ in row k of W_{k-1} where the entry is 1.

Step 3 : Put 1 in all positions (p_i, q_j) of W_k.

This procedure is known as Warshall's algorithm.

Worked Examples

W.E.1. Let $A = \{1, 2, 3, 4\}$ *and* $R = \{(1, 2), (2, 3), (3, 4), (2, 1)\}$. *Using Warshall's algorithm find the transitive closure of* R.

Solution

$$\text{Let } W_0 = M_R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

As $|A| = 4$. We have to compute W_1, W_2, W_3 and W_4. W_4 is the matrix of the transitive closure R^{∞}

To compute W_1 :

Here $k = 1$.

First let $(W_1)_{ij} = 1$ whenever $(W_0)_{ij} = 1$.

W_0 has 1's in location 2 of column 1 and location 2 of row 1. So W_1 will have 1 in position (2, 2).

$$W_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

To compute W_2 :

Here $k = 2$. Consider column 2 and row 2 of W_1. The matrix W_1 has 1's in location 1 and 2 in column 2, while it has 1's in locations 1, 2 and 3 in row 2. So W_2 will have 1 in (1, 1), (1, 2), (1, 3), (2, 1), (2, 2) and (2, 3) positions.

$$W_2 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

To compute W_3 :

Here $k = 3$. Consider column 3 of W_2 and row 3 of W_2.

The matrix W_2 has 1's in column 3 in locations 1 and 2 and it has 1 in row 3 in location 4 only. So W_3 will have 1's in (1, 4) and (2, 4) positions.

$$W_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

To compute W_4 :

Here $k = 4$. Consider column 4 of W_3 and row 4 of W_3. The matrix W_3 has 1's in column 4 in 1, 2, 3 locations, and it has no 1 in row 4. So there is no new addition of 1. So $W_4 = W_3$.

As W_4 is the matrix of R^{∞}, the relation R^{∞} is given by

$$R^{\infty} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4)\}.$$

W.E.2 *Let* A = {1, 2, 3, 4, 5} *and* R = {(1, 1), (1, 3), (1, 5), (2, 3), (2, 4), (3, 3), (3, 5), (4, 2), (4, 4), (5, 4)}. *Find the transitive closure of* R.

Solution

$$W_0 = M_R = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

k	In W_{k-1} Position of 1's in column k	Position of 1's in row k	W_k has 1's in	W_k
1	1	1,3,5	(1,1),(1,3),(1,5)	$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$
2	4	3,4	(4,3),(4,4)	$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$
3	1,2,3,4	3,5	(1,3),(1,5) (2,3),(2,5) (3,3),(3,5) (4,3),(4,5)	$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$
4	2,4,5	2,3,4,5	(2,2),(2,3),(2,4),(2,5) (4,2),(4,3),(4,4),(4,5) (5,2),(5,3),(5,4),(5,5)	$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$
5	1,2,3,4,5	2,3,4,5	(1,2),(1,3),(1,4),(1,5) (2,2),(2,3),(2,4),(2,5) (3,2),(3,3),(3,4),(3,5) (4,2),(4,3),(4,4),(4,5) (5,2),(5,3),(5,4),(5,5)	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$

$R'' = \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4), (2,5), (3,2), (3,3), (3,4), (3,5), (4,2), (4,3), (4,4), (4,5)\}$

Exercises

1. Let $A = \{1, 2, 3, 4\}$. Find the reflexive, symmetric and transitive closures of the following relations :
   a. $R = \{(1, 1), (2, 2), (3, 3)\}$
   b. $R = \{(1, 1), (2, 3), (3, 2), (4, 1)\}$
   c. $R = \{(1, 3), (1, 4), (2, 2), (3, 4), (4, 2)\}$
   d. $R = \{(1, 2), (2, 4), (3, 3), (4, 1)\}$
2. Using Warshall's algorithm find the transitive closure of the relations whose matrices are given below :

a.
$$M_R = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

b.
$$M_R = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

c.
$$M_R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

d.
$$M_R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Answers

1.a. $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$, R, R
b.i. $\{(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (3, 2), (4, 1)\}$
ii. $\{(1, 1), (2, 3), (3, 2), (4, 1), (1, 4)\}$

2. a.
$$W_4 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

b.
$$W_5 = M_{R^{\prime}} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

c.
$$W_4 = W_0 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

d.
$$W_4 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

§ 7. PARTITIONS AND EQUIVALENCE CLASSES

Consider Z, the set of all integers,

Let $A_1 = \{ x \in Z \mid x \text{ is a positive integer} \}$

$A_2 = \{ x \in Z \mid x < -100 \}$

$A_3 = \{ x \in Z \mid -100 \leq x \leq 0 \text{ and } x \text{ is divisible by } 2 \}$

$A_4 = \{ x \in Z \mid -100 \leq x \leq 0 \text{ and } x \text{ is not divisible by } 2 \}$.

Then the subsets A_1, A_2, A_3 and A_4 are mutually disjoint and their union is Z. In otherwords,

$Z = A_1 \cup A_2 \cup A_3 \cup A_4$ and $A_i \cap A_j = \phi$, $\forall\ i \neq j \in \{1, 2, 3, 4\}$. We say that $\mathbf{P} = \{A_1, A_2, A_3, A_4\}$ is a partition of the set Z.

Definition

Let X be a non-empty set. A collection **P** of non-empty subsets of X is said to be a partion of X if

i) $A \in \mathbf{P} \Rightarrow A \neq \phi$

ii) $A, B \in \mathbf{P}$ and $A \neq B \Rightarrow A \cap B = \phi$

iii) $X = \bigcup_{A \in \mathbf{P}} A$.

The members of **P** are said to be the 'blocks of the partition **P**'.

In this section we show that given an equivalence relation on a set X, we can associate a partition on X, and conversely a given partition **P** of X determines an equivalence relation on X.

EQUIVALENCE CLASSES

Let R be an equivalence relation defined on a non-empty set X. Then to $a \in X$, define a subset R(a) of X by

$$R(a) = \{ x \in X \mid (a, x) \in R \}$$

The subset R(a) is called the equivalence class of R determined by the element a. The equivalence class R(a) is also denoted by $[a]_R$.

PROPERTIES OF EQUIVALENCE CLASSES

Let R be an equivalence relation on a non-empty set X. Then

1. $R(a) \neq \phi$, for all $a \in X$.

As the relation is reflexive, $(a, a) \in R$, for all $a \in X$, hence $a \in R(a)$, for all $a \in X$. Then $R(a) \neq \phi$, for all $x \in X$.

2. $b \in R(a) \Rightarrow a \in R(b)$.

Let $a, b \in X$ such that $b \in R(a)$. Then $(a, b) \in R$. So $(b, a) \in R$, as R is symmetric. Hence $a \in R(b)$. Thus we have $b \in R(a) \Rightarrow a \in R(b)$, where $a, b \in X$.

3. $b \in R(a) \Rightarrow R(b) = R(a)$.

Let $a, b \in X$ such that $b \in R(a)$. Then $(a, b) \in R$ and hence $(b, a) \in R$. Let $c \in R(b)$. Then $(b, c) \in R$. As $(a, b) \in R$ and $(b, c) \in R$, by the transitive property of R, we have $(a, c) \in R$. So $c \in R(a)$. Thus $b \in R(a)$, and $c \in R(b) \Rightarrow c \in R(a)$. In other words, if $b \in R(a)$, then $R(b) \subseteq R(a)$. As $b \in R(a)$, $a \in R(b)$. Hence we have $R(a) \subseteq R(b)$. Thus we have $R(b) = R(a)$.

4. If $a, b \in X$, then either $R(a) \cap R(b) = \phi$ or $R(a) = R(b)$.

It is enough to show that if $R(a) \cap R(B) \neq \varphi$, then $R(a) = R(b)$. Assume that $R(a) \cap R(b) \neq \phi$. As $c \in R(a)$, by (3), we have $R(c) = R(a)$. Again as $c \in R(b)$, we have $R(c) = R(b)$. Thus $R(a) = R(c) = R(b)$. Thus if $a, b \in X$ and $R(a) \cap R(b) \neq \phi$, then $R(a) = R(b)$.

Now we have the following theorem.

Theorem 7

Every equivalence relation R on a non-empty set X determines a unique partition on X, whose "blocks" are equivalence classes of R.

Proof Let R be an equivalence relation on a non-empty set X. Let **P** be the collection of all equivalence classes of R. i.e., $\mathbf{P} = \{ R(a) \mid a \in X \}$. Then we note that

i) each element of **P** is a non-empty subset of X. (as $a \in R(a)$, for all $a \in X$).

ii) If $R(a) \neq R(b)$, then by the properties of equivalence classes, we have $R(a) \cap R(b) = \phi$.

iii) If $x \in X$, then $x \in R(x)$. So $X = \bigcup_{a \in X} R(a)$.

Thus

a. every element of **P** is a non-empty subset of X.

b. any two distinct elements of **P** are disjoint.

c. X is a union of members of **P**.

So **P** is a partition of X. As each block of this partition is of the form R(a) for same $a \in X$, each block of **P** is an equivalence class of R.

This partition **P**, whose 'blocks' are equivalence classes of R, is unique as an R - equivalence class of any element is unique. •

Remark. The unique partition **P** as X determined by an equivalence relation R on X is denoted by X | R. It is also called X modulo R.

$$X \mid R = \{ [x]_R \mid x \in X \}$$

The following theorem shows that a partition P on X determines an equivalence relation R on X.

Theorem 8

Let **P** be a partition on a non-empty set X. Then there is an equivalence relation R on X such that **P** = X | R.

Proof Let **P** be a partition on a non-empty set X. Define a relation R on X as follows:

a R b if and only if a and b are members of the same block.

i.e., a R b ⇔ a, b ∈ A for some A ∈ **P**. We claim that R is an equivalence relation

1. As $X = \bigcup_{A \in P} A$, given any a ∈ X, we can find one A ∈ **P** such that a ∈ A.

   So a R a. Thus a R a, for all a ∈ X, i.e., R is reflexive.
2. If a R b, then a, b ∈ A for some A ∈ **P**. So b R a is also true.

   ∴ a R b ⇒ b R a. Thus R is symmetric.
1. If a R b and b R c, then a, b are in some block A of **P** and b, c are in some block B of **P**. So b ∈ A ∩ B. As members of **P** are mutually disjoint, from b ∈ A ∩ B we have a = b. So a, b, c are all in the same block A of **P**. As a and c are in the same block of **P**, we have a R c. Thus a R b, b R c⇒a R c. i.e., R is transitive. We have proved that R is an equivalence relation on X. Let a ∈ X. Then a is an element of exactly one block A of P. If b ∈ A, then as a and b are in the same block of **P**, we have a R b. If b ∉ A, then there is no block of **P** containing both a and b (as blocks of **P** are mutually disjoint); so (a, b) ∉ R. Thus a R b iff b ∈ A.

   So R(a) = { b ∈ X | (a, b) ∈ R } = A. Hence the equivalence class of R are the blocks of **P** and **P** = X | R.

Worked Examples

W.E.1. Let A = {1, 2, 3, 4}. *Let* P = { {1}, {2, 4}, {3} }. *Find the equivalence relation* R *on* A *determined by the partition* **P**.

Solution

The block of **P** are {1}, {2, 4} and {3}. If R is the equivalence relation determined by **P**, then to each element a ∈ A, only the elements of the block in which a is an element, are related to a under R. So as 1 ∈ {1}, we have (1, 1) ∈ R.

As {2, 4} is a block, 2 and 4 are related only to the elements of {2, 4}.
So (2, 2), (2, 4), (4, 2), (4, 4) ∈ R. Similarly (3, 3) ∈ R.
So the equivalence relation R determined by **P** is
R = {(1, 1), (2, 2), (2, 4), (4, 2), (4, 4), (3, 3)}

W.E.2. *Let* A = {1, 2, 3, 4, 5, 6} *and* **P** = { {1, 5}, {2, 3, 6}, {4} } *be a partition on* A. *Find the equivalence relation* R *determines by* **P**.

Solution

Let R be the relation determine by the partition **P**. As {1, 5} is a block, the elements 1, 5 are related only to the elements of the subset {1, 5}. So (1, 1), (1, 5), (5, 1), (5, 5) ∈ R.

Similarly, as {2, 3, 6} is a block of **P**,
(2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (6, 2), (6, 3) and (6, 6) ∈ R
As {4} is a block, (4, 4) ∈ R.
Thus

R = {(1, 1), (1, 5), (5, 1), (5, 5)
(2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (6, 2), (6, 3), (6, 6)
(4, 4) }

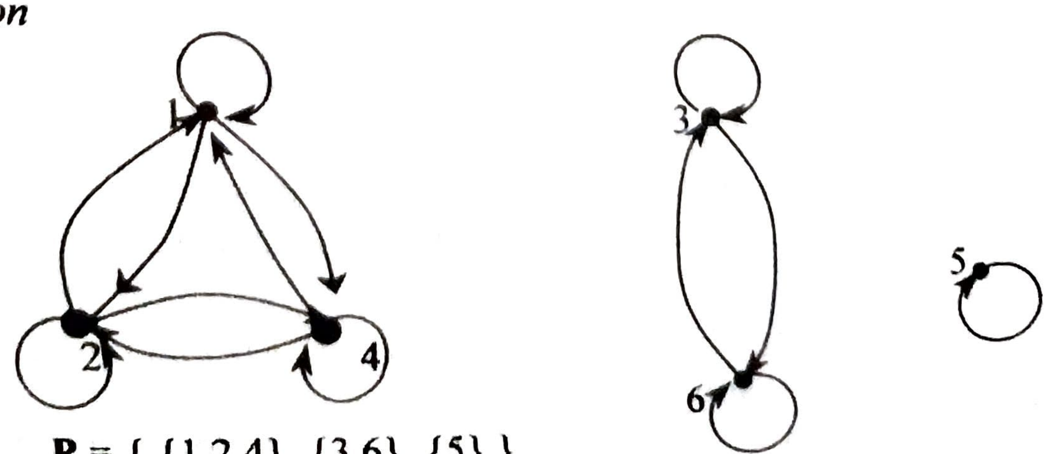
W.E.3. *Find the unique partition* **P** *on* A = {1, 2, 3, 4, 5} *determined by the equivalence relation* R, *where*
R = {(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (5, 1), (5, 3), (5, 5) }

Solution

Now R(1) = {1, 3, 5} and R(2) = {2, 4}.
Note that R(1) = R(2) = R(5) and R(2) = R(4).
The partition P is { {1, 3, 5}, {2, 4} }

W.E.4. *Find the unique partition determined by the equivalence relation* R *whose digraph is given below :*

Solution



P = { {1,2,4}, {3,6}, {5} }

W.E.5 *Let* X = {1, 2, 3, ..., 100} *and* R = {(x, y) | x − y is divisible by 7}. *Determine whether* R *is an equivalence relation or not. If it is an equivalence relation, determine the partition* X | R. *(Apr'97, B.E., M.U.)*

Solution

As x − x = 0 is divisible by 7, for all x ∈ X, (x, x) ∈ R, for all x ∈ X and hence R is reflexive.

If (x, y) ∈ R, then x − y is divisible by 7, i.e., x − y = 7m for some integer m. So y − x = 7(−m) and y − x is divisible by 7 i.e., (y, x) ∈ R. Then (x, y) ∈ R ⇒ (y, x) ∈ R and R is symmetric.

If (x, y) ∈ R and (y, z) ∈ R, then x − y = 7m and y − z = 7k for some integers m and k. Now (x − y) + (y − z) = 7(m + k). i.e., x − z = 7(m + k) and x − y is divisible by 7. Thus (x, y) ∈ R and (y, z) ∈ R ⇒ (x, z) ∈ R and so R is transitive.

As the relation R is reflexive, symmetric and transitive, it is an equivalence relation. The equivalence classes are

[1] = { 1, 8, 15, 22, 29, 36, 43, 50, 57, 64, 71, 78, 85, 92, 99 }
[2] = { 2, 9, 16, 23, 30, 37, 44, 51, 58, 65, 72, 79, 86, 93, 100 }
[3] = { 3, 10, 17, 24, 31, 38, 45, 52, 59, 66, 73, 80, 87, 94 }
[4] = { 4, 11, 18, 25, 32, 39, 46, 53, 60, 67, 74, 81, 88, 95 }
[5] = { 5, 12, 19, 26, 33, 40, 47, 54, 61, 68, 75, 82, 89, 96 }
[6] = { 6, 13, 20, 27, 34, 41, 48, 55, 62, 69, 76, 83, 90, 97 }
[7] = { 7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98 }

So **P** = X | R = { [1], [2], [3], [4], [5], [6], [7] }

Exercises

1. Let m be a positive integer. Define a relation R on N, the set of all positive integers, by x R y ⇔ x ≡ y (mod m). Find the equivalence classes of R.
2. Which of the following are partitions on A = {a, b, c}.
   i) {{a}, {b}, {c}} ii) {{a,b}, {a,c}}
   iii) {{a,b,c}} iv) {{a,b}, {c}}
   v) {{a}, {b}} vi) {{a}, {b,c}, ϕ }.
3. Find all partitions of the set A = {a,b,c}. (Dec'96, M.C.A., Bharathiar Uni.)
4. Find all partitions of the set A = {a,b,c,d}.
5. Find the equivalence relation induced by the partition {{1}, {2, 3}, {4}} of S = {1,2,3,4}. (Apr'97, B.E., Bharathi Dasan Uni.)
6. Let A = {1, 2, 3, 4} and R = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4) }. Verify that R is an equivalence relation. Determine A | R.
7. Show that an equivalence relation induces a partition and a partition induces an equivalence relation. (Dec'98, M.E., M.U.)