Set theory

Sets.

A set is any well defined list, collection or class of objects.

ex:

i) The set of numbers 1, 3, 5, 7.

ii. The set of vowels of the alphabet: a, e, i, o, u.

Null set

A set which does not contain any elements is called the empty set for null set and is denoted by ϕ.

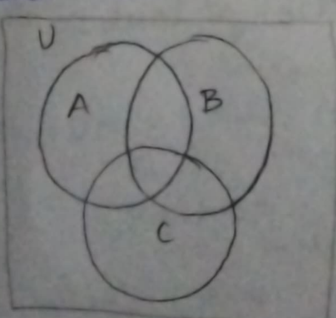
Union and Intersection.

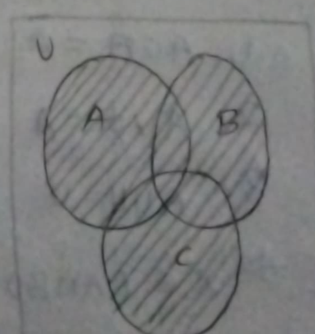
The operations of union and intersection can be defined for three or more sets in a similar manner.

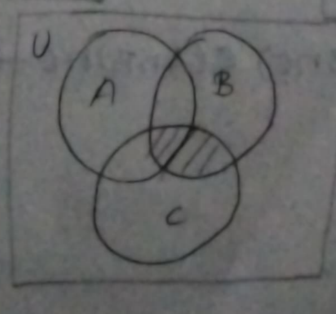
$A \cup B \cup C = \{x \mid x \in A \text{ or } x \in B \text{ (or) } x \in C\}$

$A \cap B \cap C = \{x \mid x \in A \text{ and } x \in B \text{ and } x \in C\}$

These can be illustrated, by venn diagram as follow







Example :-

Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$

Then $A \cup B \cup C = \{1, 2, 3, 4, 6, 8, 5\}$ and

$A \cap B \cap C = \{4\}$

Examples :

1. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and A the Subset $\{1, 2, 3, 4, 5, 6\}$

   Then $A' = \{7, 8, 9, 10\}$

2. Let the universal set U be the English alphabet and let $A = \{a, b, c\}$

   Then $A' = \{d, e, f, g \dots y, z\}$.

1) properties of union operation.

1. $A \cup A = A$
2. $A \cup \phi = A$
3. $A \cup U = U$

$\Rightarrow x \in (A \cap B) \cap C$

$\therefore A \cap (B \cap C) \subset (A \cap B) \cap C$ — (1)

b, let us show that $(A \cap B) \cap C \subset A \cap (B \cap C)$ $y \in (A \cap B) \cap C$

$\Rightarrow y \in (A \cap B)$ and $y \in C$

$\Rightarrow (y \in A$ and $y \in B)$ and $y \in C$

$\Rightarrow y \in A$ and $(y \in B$ and $y \in C)$

$\Rightarrow y \in A$ and $y \in (B \cap C)$

$\Rightarrow y \in A \cap (B \cap C)$

$\therefore (A \cap B) \cap C \subseteq A \cap (B \cap C)$ — (2)

from (1) and (2) we have $A \cap (B \cap C) = (A \cap B) \cap C$.

Exercise.

1, Write out the dual of each of the following expression.

i) $(A \cup B)' \cap U = (A \cap B)' \cup \phi$

ii) $(A' \cup U) \cup (A \cap U) = (A' \cap \phi) \cap (A \cup \phi)$

iii) $U \cap [(A \cup B') \cup U] = \phi \cup [(A \cap B') \cap \phi]$.

iv) $(A \cap B) \cup (C \cap \phi) \cup (U \cap A \cap B) = (A \cup B) \cap (C \cup U) \cap (\phi \cup A)$

2. Write a dual of each of the following.

i) $(B \cap C) \cup A = (B \cup A) \cap (C \cup A)$

$(B \cup C) \cap A = (B \cap A) \cup (C \cap A)$

ii) $A \cap (A' \cup B) = A \cap B$

$A \cup (A' \cap B) = A \cup B$

iii) $(A \cup \phi) \cup (U \cap A') = 0$

$(A \cap U) \cap (\phi \cup A') = \phi$

3, Prove the following identities then write out the dual Identitie and prove them.

i) $A \cap (A \cup B) = A$

$A \cup B = \{x \mid x \in A \cap x \in B)$

$A \cap B = \{x \mid x \in A \text{ and } x \in B)$

proof:

$\subseteq$: claim $A \cap (A \cup B) \subset A$

$x \in A \cap (A \cup B)$

$A = A \cap (A \cup B)$

$= (A \cap A) \cup (A \cap B)$

$= A \cup (A \cap B)$

$= A.$

Dual $A \cup (A \cap B) = A$

$A = A \cup (A \cap B)$

$= (A \cup A) \cap (A \cup B)$

$= A \cap (A \cup B)$

$= A$

Hence it is proved

4, If $A \cup B = A \cup C$, is $B = C$? Explain.

$A \cup B = A \cup C$

Need not imply $B = C$. for example

$A = \{1, 2, 3, 4\}$; $B = \{3, 5, 6\}$ and

$C = \{1, 2, 5, 6\}$

Then $A \cup B = \{1, 2, 3, 4, 5, 6\} = A \cup C$

but $B \neq C$

$\therefore A \cup B = A \cup C \Leftrightarrow B - A = C - A.$

5. ... prove the following:-

a) $A \cup (B - A) = A \cup B$.

$$A \cup (B-A) = A \cup (B \cap A')$$
$$= (A \cup B) \cap (A \cup A')$$
$$= A \cup B$$

b) $A - (B \cup C) = (A - B) \cap (A - C)$

$$= A \cap (B \cup C)'$$
$$= A \cap (B' \cap C')$$
$$= A \cap B' \cap C' \quad \text{—} ②$$

$$(A-B) \cap (A-C) = (A \cap B') \cap (A \cap C')$$

$= A \cap B'$ and $A - C = A \cap C'$.

From ① and ② we have $A - (B \cup C) = (A-B) \cap (A-C)$.

Ex:

If A and B are sets, prove that $A \subseteq B$ if and only if $B' \subseteq A'$

$A \subseteq B \Leftrightarrow$ if $x \in A$, then $x \in B$

$\rightarrow$ if $x \notin A'$, then $x \in B'$

$\rightarrow$ if $x \in B'$, then $x \in A'$

$\Rightarrow B' \subseteq A'$.

Now we consider problems on enumeration.

6. If A and B are finite sets then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Let A and B be two finite sets

If $A \cap B) = \phi$, then $n(A \cup B) = n(A) + n(B)$

$$= n(A) + n(B) - n(A \cap B)$$

$$n(A \cap B) = 0$$

Case (ii) general case

$A \cup B = A \cup (B - A)$ and as A and $(B - A)$

$$n(A \cup B) = n(A) + n(B - A) \quad -(2)$$

$B = (A \cap B) + (B - A)$ and as $A \cap B$ as are disjoint, again

from (1)

$$n(B) = n(A \cap B) + n(B - A)$$

$$\therefore n(B - A) = n(B) - n(A \cap B) \quad -(3)$$

from (2) and (3), $n(A \cup B) = n(A) + n(B)$.

7. If A, B, C are sets then show that $n(A \cup B \cup C) =$
$n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$.

$$n(A \cup B \cup C) = n(A \cup B \cup C)$$

$$= n(A \cup D)$$

$$= n(A) + n(D - n(A \cap B)$$

$$= n(A) + n(B(X - A) \cap n(B \cup C)$$

$$= n(A) + n(B \cup C) - n(A \cap B) 0$$

$$= n(A) + n(B) + n(C) - n(B \cap C) - [n(A \cap B) + n(A \cap D) - n(A \cap B) n(A \cap C)]$$

$$= n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

Rewrite the following statements using set notation.

i. x is a member of A

$x \in A$

ii. y does not belong to A

$y \in A$

iii) R is not a subset of S.

$R \subseteq S$

iv) F is a superset of G.

$F \supset G$

v) The power set of S.

2^S

Write the following sets in the set builder notation.

i) $A = \{2, 4, 6, 8, 10, \dots\}$

$A = \{x / x$ is even and positive$\}$

ii) B = {violet, indigo, blue, green, yellow, orange, red}.

$B = \{x | x$ is a colour of the

iii) $C = \{1, 3, 5, 7, 9\}$

$C = \{x | x$ is a positive odd

iv) $D = \{a, e, i, o, u\}$

$D = \{x / x$ is a vowel$\}$

3. Write all the subsets of $A = \{0, 1, 2\}$

$\phi = \{0, 1, 2\}, \{0\}, \{1,\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}$

4. How many proper nonempty subsets are there for $\{1, 2, 3, 4\}$.

It would be $24 - 2$ so the answer is

$16 - 2$ that's is 14.

5. Let $T = \{0, 1, 2, 3, 4\}$ How many subsets of T have less than 4 elements.

$\phi, \{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{0, 1\}, \{0, 2\}, \{0, 3\},$
$\{0, 4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}$
$\{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 4\}, \{0, 2, 3\}, \{0, 2, 4\}$
$\{0, 3, 4\}, \{0, 1, 2, 3\}, \{0, 1, 2, 4\}$
$\{0, 1, 2, 3, 4\}, \{0, 1, 3, 4\}$

= 25 element of subset.

6. If $A = \{1, 2, 3, 4\}$ find $P(A)$

$\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\},$
$\{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\},$
$\{2, 3, 4\}, \{1, 2, 3, 4\}$

Difference of two sets.

The difference of the sets A and B is the set of elements which is belong to A but which do not belong to B. It is denoted by A-B which is read as "A difference B" or simply

"A minus B", ie $A-B = \{x : x \in A, x \notin B\}$

A-B is also denoted by $A \setminus B$ or by $A-B$.

A-B is also known as the relative complement of B in A.

Example:

1, Let $U = \{1,2,3,4,5,6,7,8,9,10\}$ and A the subset $\{1,2,3,4,5,6\}$

Then $A' = \{7,8,9,10\}$

2, Let the universal set U be the English alphabet and let $A = \{a,b,c\}$

Then, $A' = \{d,e,f,g \ldots y,z\}$

Property of set ~~PP~~ operations.

These three main set operations are union, intersection, and complementation.

properties of union operation

1) $A \cup A = A$

2) $A \cup \phi = A$

3) $A \cup U = U$

4. $A \cup A' = U$

5. $A \subseteq A \cup B$ and $B \subset A \cup B$

6. $A \cup B = B \cup A$

7. $(A \cup B) \cup C = A \cup (B \cup C)$

properties of Intersection operation

For any three set A, B, C we have

1. $A \cap A = A$

2. $A \cap B = A$, whenever $A \subset B$. In particular $A \cap U = A$

3. $A \cap B = B$ whenever $B \subseteq A$. In particular $A \cap \phi =$

4. $A \cap B \subseteq A$ and $A \cap B \subseteq B$

5. $A \cap B = B \cap A$

6. $(A \cap B) \cap C) = A \cap (B \cap C)$ (Associative law)

Theorem 6.

For any two sets A and B

i) $(A \cup B)' = A' \cap B'$

$x \in (A \cup B)' \Leftrightarrow x \notin A \cup B$.

$\Leftrightarrow x \in A$ and $x \in B$

$\Leftrightarrow x \in A'$ and $x \in B'$

$\Leftrightarrow x \in A' \cap B'$

Thus $(A \cup B)' = A' \cap B'$

ii) $(A \cap B)' = A' \cup B'$

$y \in (A \cap B)' \Leftrightarrow y \in A \cap B$

$\Leftrightarrow$ either $y \in A$ or $y \in B$

$\Leftrightarrow$ either $y \in A'$ or $y \in B'$

$\Leftrightarrow y \in A' \cup B'$

Thus we have $(A \cap B)' = A' \cup B'$

ex:

write $A \cup (B \cap (C \cap (D \cup E))$

i) without using $\cup$. ii) without using $\cap$.

i) $A \cup (B \cap C \cap (D \cup E))$

$= A \cup (B \cap (C \cap (D' \cap E')'))$

$= (A' \cap (B \cap C \cap (D' \cap E')')')'$

ii) $A \cup (B \cap C \cap (D \cup E))$

$= A \cup (B' \cup C' \cup (D \cup E)')')$

Properties of the difference operations

Let, A, B, C are sets and U be the universal set

Then,

1, $A' = U - A$

2, $A - B = A \cap B'$

3, $A - A = \phi$

4, $A - \phi = A$

5, $A - B = B - A$

6, $A - B = A$

7, $A - B = \phi$

proof:

we prove the properties (2) (6), and (7)

2, $x \in A - B \Leftrightarrow x \in A$ but $x \notin B$

$\Leftrightarrow x \in A$ and $x \in B'$

$\Leftrightarrow x \in A \cap B'$

So, $A - B = A \cap B'$

6, Let $A - B = A$

Then $x \in A \Rightarrow x \in A - B \Rightarrow x \in A$ but

$x \notin B$

$\Rightarrow x \notin B$

So $x \in A \Rightarrow x \notin B$ and so $A - B = \phi$

Conversely let $A \cap B = \phi$

Then $x \in A \Rightarrow x \notin B$ as $A \cap B = \phi$

So, $x \in A \Rightarrow x \in A$ and $x \notin B$.

$\Rightarrow x \in A - B$

So, $A \subset A - B$.

Thus, $A = A - B$ So, $A - B \subset A$

7, $A-B=\phi \Leftrightarrow A\cap B'=\phi$ (by 2)

$\Leftrightarrow$ if $x\in A$, then $x\notin B'$

$\Leftrightarrow$ if $x\in A$, then $x\notin B$

$\Leftrightarrow$ if $A\subseteq B$.

Verification of the basic laws of algebra of sets by venn diagrams.

In earlier sections we have proved some basic law of set algebra we list those laws in the table. we use the symbol U to denote the universal set.

The basic laws of sets Algebra.

Commutative laws $\rightarrow$ $A\cap B = B\cap A$

$A\cup B = B\cup A$

Associative law $\rightarrow$ $A\cup(B\cup C) = (A\cup B)\cup C$

$A\cap(B\cap C) = (A\cap B)\cap C$

Distributive law $\rightarrow$ $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

$A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

Absorption law $\rightarrow$ $A\cup(A\cap B) = A$

$A\cap(A\cup B) = A$

De morgan's law $\rightarrow$ $(A\cup B)' = A'\cap B'$

$(A\cup B)' = A'\cup B'$

Complement law $\rightarrow$ $\phi' = U$; $U' = \phi$; $(A')' = A$

$A\cap A' = \phi$; $A\cup A' = U$.

Example:

1) Show that $(A-B)-C = A-(B\cup C)$

$$(A-B)-C = (A\cap B')\cap C'$$
$$= A\cap(B'\cap C')$$
$$= A\cap(B\cup C)'$$
$$= A-(B\cup C)$$

2. Show that $A-(B-C) = (A-B)\cup(A\cap C)$

$$A-(B-C) = A-(B\cap C')$$
$$= A\cap(B\cap C')'$$
$$= A\cap(B'\cup C)$$
$$= (A\cap B')\cup(A\cap C)$$
$$= (A-B)\cup(A\cap C)$$

3. Show that $A\cap(B-C) = (A\cap B)-(A\cap C)$

$$A\cap(B-C) = (A\cap B)-(A\cap C)$$
$$= (A\cap B)-(A\cap C)$$
$$= (A\cap B)\cap(A'\cap C')$$
$$= (A\cap B\cap A')\cup(A\cap B\cap C')$$
$$= (\phi\cap B)\cup(A\cap B\cap C')$$
$$= \phi\cup(A\cap B\cap C')$$
$$= A\cap B\cap C'.$$
$$= A\cap(B-C) = LHS.$$

4. Show that $A \cup (B-C) = (A \cup B) - (C-A)$

$A \cup (B-C) = A \cup (B \cap C')$

$= (A \cup B) \cap (A \cup C')$

$= (A \cup B) \cap (A' \cap C')'$

$= (A \cup B) \cap (C \cap A')'$

$= (A \cup B) \cap (C-A)'$

$= (A \cup B) - (C-A)$

Note

$A \cup (B-C) \neq (A \cup B) - (A \cup C)$

ie. union is not distributive over difference.

5. If $A \cup B = A \cup C$ and $A \cap B = A \cap C$.

prove that $B = C$

As it is given that $A \cup B = A \cup C$, we have

$B \cap (A \cup B) = (B \cap (A \cup C)$ — (1)

LHS $= B \cap (A \cup B) = B$ as $B \subseteq A \cup B \rightarrow$ (2)

RHS $= B \cap (A \cup C)$

$= (B \cap A) \cup (B \cap C)$ ' Distributive law.

$= (A \cap B) \cup (B \cap C)$ (commutative

$= (A \cap C) \cup (B \cap C)$ since $A \cap B = A \cap C$

$= (A \cup B) \cap C)$

$= (A \cup C) \cap C)$

$= C$.

6. If $A \cup B = A \cup C$, is $B = C$? Explain

$$A \cup B = A \cup C$$

Need not imply $B = C$

$A = \{1, 2, 3, 4\}$; $B = \{3, 5, 6\}$ and $C = \{1, 2, 5, 6\}$

$A \cup B = \{1, 2, 3, 4, 5, 6\} = A \cup C$

but $B \neq C$

$(\because A \cup B = A \cup C \Leftrightarrow B - A = C - A)$