UNIT -2
MATRICES

A matrix is defined to be a rectangular array of numbers arranged into rows and columns. It is written as follows:-

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots\ldots & a_{3n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots\ldots & a_{mn} \end{bmatrix}$$

Special Types of Matrices:

(i) A **row matrix** is a matrix with only one row. E.g., [2 1 3].

(ii) A **column matrix** is a matrix with only one column. E.g., $\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$.

(iii) **Square matrix** is one in which the number of rows is equal to the number of columns.

If A is the square matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots\ldots & a_{3n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots\ldots & a_{mn} \end{bmatrix}$$

then the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots\ldots & a_{3n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots\ldots & a_{mn} \end{vmatrix}$$

is called the determinant of the matrix A and it is denoted by |A| or detA.

(iv) **Scalar matrix** is a diagonal matrix in which all the elements along the main diagonal are equal.

E.g., $\begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_1 \end{bmatrix}$

(v) **Unit matrix** is a scalar matrix in which all the elements along the main diagonal are unity.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(vi) **Null or Zero matrix**. If all the elements in a matrix are zeros, it is called a null or zero matrix and is denoted by 0.

(vii) **Transpose matrix**. If the rows and columns are interchanged in matrix A, we obtain a second matrix that is called the transpose of the original matrix and is denoted by A^t.

(viii) **Addition of matrices.** Matrices are added, by adding together corresponding elements of the matrices. Hence only matrices of the same order may be added together. The result of addition of two matrices is a matrix of the same order whose elements are the sum of the same elements of the corresponding positions in the original matrices.

E.g., $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{bmatrix} = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \\ a_5+b_5 & a_6+b_6 \end{bmatrix}$

Problem:

Given $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ 5 & 0 & 6 \end{bmatrix}$; $B = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$; compute 3A-4B

Solution :

$$3A-4B=3\begin{bmatrix}1&0&2\\3&1&4\\5&0&6\end{bmatrix}-4\begin{bmatrix}2&1&-1\\3&0&-2\\0&1&1\end{bmatrix}$$

$$=\begin{bmatrix}3&0&6\\9&3&12\\15&0&18\end{bmatrix}-\begin{bmatrix}8&4&-4\\12&0&-8\\0&4&4\end{bmatrix}$$

$$=\begin{bmatrix}-5&-4&10\\-3&3&20\\15&-4&14\end{bmatrix}$$

Problem: Find values of $x, y, z\ and\ \omega$ that satisfy the matrix relationship

$$\begin{bmatrix}x+3&2y+5\\z+4&4x+5\\\omega-2&3\omega+1\end{bmatrix}=\begin{bmatrix}1&-5\\-4&2x+1\\2\omega+5&-20\end{bmatrix}$$

Solution :

From the equality of these two matices we get the equations

$$x+3=1 \qquad 4x+5=2x+1$$
$$2y+5=-5 \qquad \omega-2=2\omega+5$$
$$z+4=-4 \qquad 3\omega+1=-20$$

Solving these equations we get

$x=-2, y=-5,\ z=-8,\ \omega=-7$

Multiplication of Matrices.

If A is a m × n matrix with rows $A_1, A_2, \ldots\ldots, A_m$ and B is a n × p matrix with columns $B_1, B_2, \ldots.., B_p$, then the prodduct AB is a m × p matrix C whose elements are given by the formula $C_{ij} = A_i . B_j$.

$$\text{Hence } C = AB = \begin{bmatrix}A_1.B_1 & A_1.B_2 & \cdots & A_1.B_p\\ A_2.B_1 & A_2.B_2 & \cdots & A_2.B_p\\ \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ A_m.B_1 & A_m.B_2 & \cdots & A_m.B_p\end{bmatrix}$$

Inverse of a Matrix

Problem: Find the inverse of the matrix $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix}$.

Solution:

$$\det\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix} = 2\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix}$$

= 2(1 + 3) – 1(-6) – 1(-2)

= 8 + 6 + 2

= 16.

Form the matrix of minor determinants:

$$\begin{pmatrix} \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 4 & -6 & -2 \\ 0 & 4 & -4 \\ 4 & 6 & 2 \end{pmatrix}.$$

Adjust the signs of every other element (starting with the second entry):

$$\begin{pmatrix} 4 & 6 & -2 \\ 0 & 4 & 4 \\ 4 & -6 & 2 \end{pmatrix}$$

Take the transpose and divide by the determinant:

$$\frac{1}{16}\begin{pmatrix} 4 & 0 & 4 \\ 6 & 4 & -6 \\ -2 & 4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{3}{8} & \frac{1}{4} & -\frac{3}{8} \\ -\frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{pmatrix}$$

So the inverse matrix is $\begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{3}{8} & \frac{1}{4} & -\frac{3}{8} \\ -\frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{pmatrix}$.

Problem: Show that A = $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ satisfies the equation $A^2 - 4A - 5I = 0$. Hence determine its inverse.

Solution: $A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$

$$4A = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$5I = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $A^2 - 4A - 5I = 0$.

Multiplying by A^{-1}, we have

$A^{-1}A^2 - 4A^{-1}A - 5A^{-1}I = 0$

i.e., $A - 4I - 5A^{-1} = 0$

Therefore $5A^{-1} = A - 4I$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

Therefore $A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$.

Rank of a Matrix

A sub-matrix of a given matrix A is defined to be either A itself or an array remaining after certain rows and columns are deleted from A.

The determinants of the square sub-matrices are called the minors of A.

The rank of an $m \times n$ matrix A is r iff every minor in A of order r + 1 vanishes while there is at least one minor of order r which does not vanish.

Problem: Find the rank of the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 6 & 3 \\ 3 & 13 & 4 \end{bmatrix}$.

Solution:

Minor of third order = $\begin{vmatrix} 1 & -1 & 2 \\ 2 & 6 & 3 \\ 3 & 13 & 4 \end{vmatrix}$

= 0.

The minors of order 2 are obtained by deleting any one row and any one column.

One of the minors of orders 2 is $\begin{vmatrix} 1 & -1 \\ 2 & 6 \end{vmatrix}$

Its value is 8.

Hence the rank of the given matrix is 2.

Rank of a Matrix by Elementary Transformations:

Problem: Find the rank of the matrix A = $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$.

Solution: The given matrix is

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 6 & 3 \\ 3 & 13 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & -6 \\ 0 & -1 & -8 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 6 \\ 0 & -1 & -8 \end{bmatrix} R_2 \to R_2(-1)$$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix} R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{matrix} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 - 5C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} C_3 \to C_3 - 6C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_3 \to \frac{R_3}{-2}$$

Hence A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which is a unit matrix of order 3.

Hence the rank of the given matrix is 3.

Procedure for finding the solutions of a system of equations:

Let the given system of linear equations be

$a_{11} x_1 + a_{12} x_2 + \ldots\ldots + a_{1n} x_n = b_1$

$a_{21} x_1 + a_{22} x_2 + \ldots\ldots + a_{2n} x_n = b_2$

....

$a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots + a_{mn} x_n = b_m$

Step 1: Construct the coefficient matrix which is denoted by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots\ldots & a_{3n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots\ldots & a_{mn} \end{bmatrix}$$

Step 2: Construct the augmented matrix which is denoted by [A, B]

$$[A, B] = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} & b_2 \\ a_{31} & a_{32} & \ldots\ldots & a_{3n} & b_3 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} & b_m \end{bmatrix}$$

Step 3: Find the ranks of both the coefficient matrix and augmented matrix which are denoted by R(A) and R(A, B).

Step 4: Compare the ranks of R (A) and R(A, B) we have the following results.

(a) If R(A) = R(A, B) = n (number of unknowns) then the given system of equations are consistent and have unique solutions.

(b) If R(A) = R(A, B) < n (number of unknowns) then the given system of equations are consistent and have infinite number of solutions.

(c) If R(A) ≠ R(A, B) then the given system of equations are inconsistent (that is the given system of equations have no solution).

Problem: Test for consistency and hence solve $x-2y+3z=2, 2x-3z=3, x+y+z=0.$

Solution: The coefficient matrix

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & -3 \\ 1 & 1 & 1 \end{bmatrix}$$

The augmented matrix

$$[A, B] \sim \begin{bmatrix} 1 & -2 & 3 & 2 \\ 2 & 0 & -3 & 3 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & -9 & -1 \\ 0 & -3 & -2 & -2 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & \frac{-9}{4} & \frac{-1}{4} \\ 0 & -3 & -2 & -2 \end{bmatrix} R_2 \to \frac{R_2}{4}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & \frac{-9}{4} & \frac{-1}{4} \\ 0 & -3 & -2 & -2 \end{bmatrix} R_2 \to \frac{R_2}{4}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & \frac{-9}{4} & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{-5}{19} \end{bmatrix} R_3 \to \frac{4R_3}{19}$$

Here rank of coefficient matrix is 3.

Rank of augmented matrix is 3.

Hence the given system of equations are consistent and have unique solution.

Problem: Test the consistency of the following system of equations and if consistent solve

$2x - y - z = 2, x + 2y + z = 2, 4x - 7y - 5z = 2.$

Solution:

The coefficient matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{bmatrix}$$

The augmented matrix

$$[A, B] \sim \begin{bmatrix} 2 & -1 & -1 & 2 \\ 1 & 2 & 1 & 2 \\ 4 & -7 & -5 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & -1 & -1 & 2 \\ 4 & -7 & -5 & 2 \end{bmatrix} R_1 \sim R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & -15 & -9 & -6 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & 5 & 3 & 2 \end{bmatrix} R_3 \to R_3 + R_2$$

Here rank of coefficient matrix is R(A) = 2.

Rank of augmented matrix is R(A, B) = 2.

i.e., R(A) = R(A, B) < 3 (the number of unknowns)

Hence the given system of equations are consistent but have infinite number of solutions. Here the reduced system is

5y +3z = 2

x + 2y + z = 2

i.e., $y = \frac{2-3z}{5}$

$x = 2 - z - 2\left(\frac{2-3z}{5}\right)$

$= \frac{6+z}{5}$

i.e., $x = \frac{6+k}{5}$, $y = \frac{2-3z}{5}$, z = k where z = k is the parameter.

Solution of Simultaneous Equations

Problem: Solve the system of equations
$$\begin{aligned} 2x + y + z &= 6 \\ x + 2y + 3z &= 6.5 \\ 4x - 2y - 5z &= 2 \end{aligned}$$

Solution:

It can be represented as: $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 4 & -2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 6.5 \\ 2 \end{pmatrix}$.

To see whether a solution exists we need to find $\det\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 4 & -2 & -5 \end{pmatrix}$.

This determinant is $2\begin{vmatrix} 2 & 3 \\ -2 & -5 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 4 & -5 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix}$ = 2(-4) – (-17) + (-10) = -1

Therefore we know that the equations do have a unique solution.

To find the solution we need to find the inverse of the matrix $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 4 & -2 & -5 \end{pmatrix}$.

Find the determinant: we have already found that this is -1.

Form the matrix of minor determinants (which, for a particular entry in the matrix, is the determinant of the 2 by 2 matrix that is left when the row and column containing the entry are deleted):

$$\begin{pmatrix} -4 & -17 & -10 \\ -3 & -14 & -8 \\ 1 & 5 & 3 \end{pmatrix}$$

Adjust the signs of every other element (starting with the second entry):

$$\begin{pmatrix} -4 & 17 & -10 \\ 3 & -14 & 8 \\ 1 & -5 & 3 \end{pmatrix}$$

Take the transpose and divide by the determinant:

$$\frac{1}{-1}\begin{pmatrix} -4 & 3 & 1 \\ 17 & -14 & -5 \\ -10 & 8 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -3 & -1 \\ -17 & 14 & 5 \\ 10 & -8 & -3 \end{pmatrix}$$

So the inverse matrix is $\begin{pmatrix} 4 & -3 & -1 \\ -17 & 14 & 5 \\ 10 & -8 & -3 \end{pmatrix}$.

Hence the solutions to the equations are found by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 & -3 & -1 \\ -17 & 14 & 5 \\ 10 & -8 & -3 \end{pmatrix}\begin{pmatrix} 6 \\ 6.5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2.5 \\ -1 \\ 2 \end{pmatrix}.$$

Therefore x = 2.5, y = -1 and z = 2.

Cayley – Hamilton theorem:

Every square matrix satisfies its own characteristic equation.

Problem: *Verify Cayley – Hamilton theorem for the matrix* $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ *and hence find the inverse of A.*

Solution : The characteristic equation of matrix A is

$\lambda^3 - \lambda^2(1+4+6) + \lambda(-1-3+0) - [1(-1)-2(-3)+3(-2)] = 0$

$\lambda^3 - 11\lambda^2 - 4\lambda + 1 = 0$, which is the characteristic equation.

By Cayley – Hamilton theorem , we have to prove

$A^3 - 11A^2 - 4A + 1 = 0$

$$A^2 = A \times .A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{pmatrix}$$

$$A^3 = A^2 \times A = \begin{pmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{pmatrix}$$

$$A^3 - 11A^2 - 4A + I = \begin{pmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{pmatrix} - 11 \begin{pmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{pmatrix} - 4 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Hence the theorem is verified.

To find A^{-1}

We have $A^3 - 11A^2 - 4A + I = 0$

$I = -A^3 + 11A^2 + 4A$

$A^{-1} = -A^2 - 11A + 4I$

$$= -\begin{pmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{pmatrix} - 11 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} + 4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \mathbf{A^{-1}} = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

Problem: Find all the eigen values and eigen vectors of the matrix A = $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$

Solution : Given A = $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$

The characteristic equation of the matrix is

$\lambda^3 - \lambda^2(2+1+1) + \lambda(-3+1+1) - [2(-3)-1(-1)-1(-1)] = 0$

$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$, which is the characteristic equation.

1	1	-4	-1	4
	0	1	-3	-4
	1	-3	-4	0

$\lambda = 1$ is a root.

The other roots are $\lambda^2 - 3\lambda - 4 = 0$

$\Rightarrow (\lambda - 4)(\lambda + 1) = 0$

$\Rightarrow \lambda = 4 , -1$

Hence **λ = 1 , 4 , -4.**

The eigen vectors of the matrix A is given by $(A - \lambda I)X = 0$

i.e. $\begin{pmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$$\left.\begin{aligned} (2-\lambda)x_1 + x_2 - x_3 &= 0 \\ x_1 + (1-\lambda)x_2 - 2x_3 &= 0 \\ -x_1 - 2x_2 + (1-\lambda)x_3 &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

When $\lambda = 1$, equation (1) becomes

$x_1 + x_2 - x_3 = 0$

$x_1 + 0x_2 - 2x_3 = 0$

$-x_1 - 2x_2 + 0x_3 = 0$

Take first and second equation,

$x_1 + x_2 - x_3 = 0$

$x_1 + 0x_2 - 2x_3 = 0$

$\Rightarrow \frac{x_1}{-2+0} = \frac{-x_2}{-2+1} = \frac{x_3}{0-1}$

$\Rightarrow \frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{-1}$

$\therefore$ **x_1 =** $\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$

When λ = -1 , equation (1) becomes

$3x_1+x_2-x_3 = 0$

$x_1+2x_2-2x_3 = 0$

$\Rightarrow \frac{x_1}{-2+2} = \frac{-x_2}{-6+1} = \frac{x_3}{6-1}$

$\Rightarrow \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$

$\therefore$ **x_2=** $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

When λ = 4 , equation (1) becomes

$-2x_1+x_2-x_3 = 0$

$x_1-3x_2-2x_3 = 0$

$\Rightarrow \frac{x_1}{-2-3} = \frac{-x_2}{4+1} = \frac{x_3}{6-1}$

$\Rightarrow \frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{1}$

$\therefore$ **x_3=** $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$. Hence **Eigen vector =** $\begin{pmatrix} -2 & 0 & -1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$

Problem: Find all the eigen values and eigen vectors of $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

Solution : Given A = $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

The characteristic equation of the matrix is

$\lambda^3 - \lambda^2(2+3+2) + \lambda(4+3+4) - [2(4)-2(1)+1(-1)] = 0$

$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$, which is the characteristic equation.

1	1	-7	11	-5
	0	1	-6	5
	1	-6	5	0

$\lambda = 1$ is a root.

The other roots are $\lambda^2 - 6\lambda + 5 = 0$

$\Rightarrow (\lambda - 1)(\lambda - 5) = 0$

$\Rightarrow \lambda = 1, 5$

Hence λ = 1 , 1 , 5.

The eigen vectors of the matrix A is given by $(A - \lambda I)X = 0$

i.e. $\begin{pmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$$\left.\begin{aligned} (2-\lambda)x_1 + 2x_2 + x_3 &= 0 \\ x_1 + (3-\lambda)x_2 + x_3 &= 0 \\ x_1 + 2x_2 + (2-\lambda)x_3 &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

When $\lambda = 1$, equation (1) becomes

$x_1 + 2x_2 + x_3 = 0$

$x_1 + 2x_2 + x_3 = 0$

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Here all the equations are same.

Put $x_3 = 0$, we get $x_1 + 2x_2 = 0$

$x_1 = -2x_2$

$$\Rightarrow \frac{x_1}{-2} = \frac{x_2}{1}$$

$$\therefore \mathbf{x_1} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda = 1$, put $x_2 = 0$,we get

$x_1+x_3 = 0$

$x_1= -x_3$

$$\Rightarrow \frac{x_1}{-2} = \frac{x_2}{1}$$

$$\therefore \mathbf{x_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

When $\lambda = 5$, equation(1) becomes

$-3x_1+2x_2+x_3 = 0$

$x_1-2x_2+x_3 = 0$ (taking first and second equation)

$$\Rightarrow \frac{x_1}{2+2} = \frac{-x_2}{-3-1} = \frac{x_3}{6-2}$$

$$\Rightarrow \frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4}$$

$\therefore$ **x_3=** $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. **Hence Eigen vector =** $\begin{pmatrix} -2 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Problem: ***Find the eigen values and eigen vectors of*** $\begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

Solution : The characteristic equation of matrix A is

$$\lambda^3 - \lambda^2(1+2-1) + \lambda(-3-1+3) - [1(-3) - 1(1) - 2(-1)] = 0$$

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

2	1	-2	-1	2
	0	2	0	-2
	1	0	-1	0

λ = 2 is a root.

The other roots are

$$\lambda^2 - 1 = 0$$

$$(\lambda - 1)(\lambda + 1) = 0$$

$$\lambda = 1, -1$$

Hence **λ = 2 , 1 , -1**

The eigen vectors of matrix A is given by

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\left.\begin{aligned} (1-\lambda)x_1 + x_2 - 2x_3 &= 0 \\ -x_1 + (2-\lambda)x_2 + x_3 &= 0 \\ 0x_1 + x_2 + (-1-\lambda)x_3 &= 0 \end{aligned}\right\} \quad \text{.............................(1)}$$

When λ = 1 ,Equation (1) becomes

$$0x_1 + x_2 - 2x_3 = 0$$

$$-x_1 + x_2 + x_3 = 0$$

$$\Rightarrow \frac{x_1}{1+2} = \frac{-x_2}{0-2} = \frac{x_3}{0+1}$$

$$\Rightarrow \frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$\therefore X_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

When $\lambda = -1$,Equation (1) becomes

$x_2=0$

$2x_1-2x_3 = 0$

$x_1 = x_3$

$$X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

When $\lambda = 2$,

Equation (1) becomes

$-x_1+x_2-2x_3=0$

$-x_1+0x_2+x_3=0$(taking first and second equation)

$$\Rightarrow \frac{x_1}{1-0} = \frac{-x_2}{-1-2} = \frac{x_3}{0-1}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{3} = \frac{x_3}{1}$$

$$\therefore X_3 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

Hence **Eigen vector =** $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix}$