UNIT-III: TEST OF SIGNIFICANCE

Unit-III	Test of Significance / Testing of Hypothesis
3.1	Sampling Distribution
3.2	Standard Error
3.3	Test of Significance
3.4	Null and Alternative Hypotheses
3.5	Type I and Type II Errors
3.6	Large Sample Tests
3.7	Test for Single Mean
3.8	Difference of Means
3.9	Single Proportion
3.10	Difference of Proportions

3.0 Introduction:

It is not easy to collect all the information about population and also it is not possible to study the characteristics of the entire population (finite or infinite) due to time factor, cost factor and other constraints. Thus, we need sample. Sample is a finite subset of statistical individuals in a population and the number of individuals in a sample is called the samplesize.

Sampling is quite often used in our day-to-day practical life. For example, in a shop we assess the quality of rice, wheat or any other commodity by taking a handful of it from the bag and then to decide to purchase it or not.

Parameter(s) and Statistic(s):

The statistical constants of the **population** {such as mean, (μ), variance (σ^2), correlation coefficient (ρ) and proportion (P)} are called **'Parameters'**.

The statistical constants **computed from the samples** {corresponding to the parameters namely mean ($\bar{x}$), variance (s^2), sample correlation coefficient (r) and proportion (p) etc,} are called **Statistics**.

Parameters are functions of the population values while statistic are functions of the sample observations. In general, population parameters are unknown and sample statistics are used as the estimates of the population parameters.

3.1 Sampling Distribution:

The distribution of values of the statistic, calculated from all possible samples taken from the population, is called sampling distribution of the statistic.

The distribution of all possible values which can be assumed by some statistic measured from samples of same size 'n' randomly drawn from the same population of size N, **is called as sampling distribution of the statistic** (DANIEL and FERREL).

Consider a population with 'N' values. Let us take a random sample of size n from this population, then there are $NCn = \frac{N!}{n!.(N-n)!}$ = k (say), possible samples. From each of these k samples if we compute a statistic (e.g mean, variance, correlation coefficient, skewness etc.) and then we form a frequency distribution for these k values of a statistic. Such a distribution is called sampling distribution of that statistic.

For example, we can compute some statistic $t = t(x_1, x_2, \ldots, x_n)$ for each of these k samples. Then $t_1, t_2, \ldots, t_k$ determine the sampling distribution of the statistic 't'. In other words, the statistic 't' may be regarded as a random variable which can take the values $t_1, t_2, \ldots, t_k$ and we can compute various statistical constants like mean, variance, skewness, kurtosis etc., for this sampling distribution.

The mean of the sampling distribution t is $\bar{t} = \frac{1}{K}[t_1 + t_2 + \ldots\ldots + t_k] = \frac{1}{K}\sum_{i=1}^{k} t_i$

and $var(t) = \frac{1}{K}[(t_1 - \bar{t})^2 + (t_2 - \bar{t})^2 + \ldots\ldots + (t_k - \bar{t})^2]$

$$= \frac{1}{K}\Sigma(t_i - \bar{t})^2$$

3.2 Standard Error:

The standard deviation of the sampling distribution of a statistic is known as its standard error. It is abbreviated as S.E. For example, the standard deviation of the sampling distribution of the mean $\bar{x}$ known as the standard error of the mean,

Where $v(\bar{x}) = v\left(\frac{x_1 + x_2 + \ldots\ldots x_n}{n}\right)$

$$= \frac{v(x_1)}{n^2} + \frac{v(x_2)}{n^2} + \ldots\ldots + \frac{v(x_n)}{n^2}$$

$$= \frac{\sigma^2}{n^2} + \frac{\sigma^2}{n^2} + \ldots\ldots + \frac{\sigma^2}{n^2} = \frac{n\sigma^2}{n^2}$$

The S.E of the mean is $\frac{\sigma}{\sqrt{n}}$

Uses of standard error

i) Standard error plays a very important role in the large sample theory and forms the basis of the testing of hypothesis.

ii) The magnitude of the S.E gives an index of the precision of the estimate of the parameter.

iii) The reciprocal of the S.E is taken as the measure of reliability or precision of the sample.

iv) S.E enables us to determine the probable limits within which the population parameter may be expected to lie.

The standard errors of the some of the well-known statistic for large samples are given below, where n is the sample size, σ^2 is the population variance and P is the population proportion and Q = 1-P. n_1 and n_2 represent the sizes of two independent random samples respectively.

Sl.No.	Statistic	Standard Error
1.	Sample mean $\bar{x}$	$\frac{\sigma}{\sqrt{n}}$
2.	Observed sample proportion p	$\sqrt{\frac{PQ}{n}}$
3.	Difference of two samples means $(\bar{x}_1 - \bar{x}_2)$	$\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$
4.	Difference of two sample proportions $p_1 - p_2$	$\sqrt{\frac{P_1Q_1}{n_1} + \frac{P_2Q_2}{n_2}}$

3.3 Tests of Significance

TOS is a test procedure used to test the significance of the difference between the **statistic and the parameter** values.

Test Procedure:

Steps involved in the test procedure for testing hypothesis (Tests of Significance) is given below. (for both large sample and small sample tests)

1. Null hypothesis: set up null hypothesis H_0.
2. Alternative Hypothesis: Set up alternative hypothesis H_1, which is complementary to H_0 which will indicate whether one tailed (right or left tailed) or two tailed test is to be applied.
3. Level of Significance: Choose an appropriate level of significance (α), α is fixed in advance.
4. Test statistic (or test of criterion):

   Calculate the value of the test statistic, $Z = \frac{t - E(t)}{S.E.(t)}$ under the null hypothesis, where t is the sample statistic.

5. Critical Value: Z_α
6. Inference: We compare z the computed value of Z (in absolute value) with the significant value (critical value) $Z_{\alpha/2}$ (or Z_α). If $|Z| > Z_\alpha$, we reject the null hypothesis H_0 at α % level of significance and if $|Z| \le Z_\alpha$, we accept H_0 at α % level of significance.

Note:

1. **Large Sample**: A sample is large when it consists of **more than 30 (n>30)** items.
2. **Small Sample**: A sample is small when it consists of **30 or less than 30 ($n \le 30$)** items.

3.4 Null Hypothesis and Alternative Hypothesis

A **hypothesis** is an **assumption** about the **population Parameter.**

Hypothesis testing begins with an **assumption** called a Hypothesis, that we make about a population parameter.

The conventional approach to hypothesis testing is not to construct a single hypothesis about the population parameter, but rather to set up two different hypotheses so that of one hypothesis is accepted, the other is rejected and vice versa.

Null Hypothesis:

A hypothesis **of no difference** is called **null hypothesis** and is usually denoted by H_0 "Null hypothesis is the hypothesis" which is tested for possible rejection under the assumption that it is true by Prof. R.A. Fisher. It is very useful tool in test of significance. For example: If we want to test whether **the special classes (for Hr. Sec. Students) after school hours has benefited the students or not.** We shall set up a null hypothesis that **"H_0: special classes after school hours has not benefited the students"**.

Alternative Hypothesis:

Any hypothesis, which is **complementary to the null hypothesis**, is called an alternative hypothesis, usually denoted by H_1. For example, if we want to test the null hypothesis that the population has a specified mean μ_0 (say),

i.e., : Step 1: null hypothesis H_0: $\mu = \mu_0$

2. Alternative hypothesis may be
   i) $H_1 : \mu \neq \mu_0$ (ie., $\mu > \mu_0$ or $\mu < \mu_0$) → **two – tailed alternative**
   ii) $H_1 : \mu > \mu_0$ → **right-tailed alternative**
   iii) $H_1 : \mu < \mu_0$ → **left –tailed alternative**

The setting of alternative hypothesis is very important, since, it enables us to decide whether we have to use a single-tailed (right or left) or two tailed test.

3.5 Type I and Type II Errors

In practice, type I error amounts to rejecting a lot when it is good and type II error may be regarded as accepting the lot when it is bad. Thus, we find ourselves in the situation which is described in the following table.

Obviously, the first two possibilities lead to errors. In a statistical hypothesis testing experiment, a Type I error is committed by rejecting the null hypothesis when it is true. On the other hand, a Type II error is committed by not rejecting (accepting) the null hypothesis when it is false.

Originally ↓ \ Decision→	Accept H_0	Reject H_0
H_0 is true	Correct decision *Accepting H_0, while it is true*	Type I Error *Rejecting H_0, while it is true*
H_0 is false	Type II error *Accepting H_0, while it is false*	Correct decision *Rejecting H_0, while it is false*

When a statistical hypothesis is tested, there are four possibilities.

i. The hypothesis is true but our test rejects it (Type I error)
ii. The hypothesis is false but our test accepts it (Type II error)
iii. The hypothesis is true and our test accepts it (correct decision)
iv. The hypothesis is false and our test rejects it (correct decision)

ie., α = P (Type I error) = P (rejecting H_0 | H_0 is true)

β = P (Type II error) = P (Accepting H_0 | H_0 is false)

3.6 Level of significance and Critical value:

Level of significance:

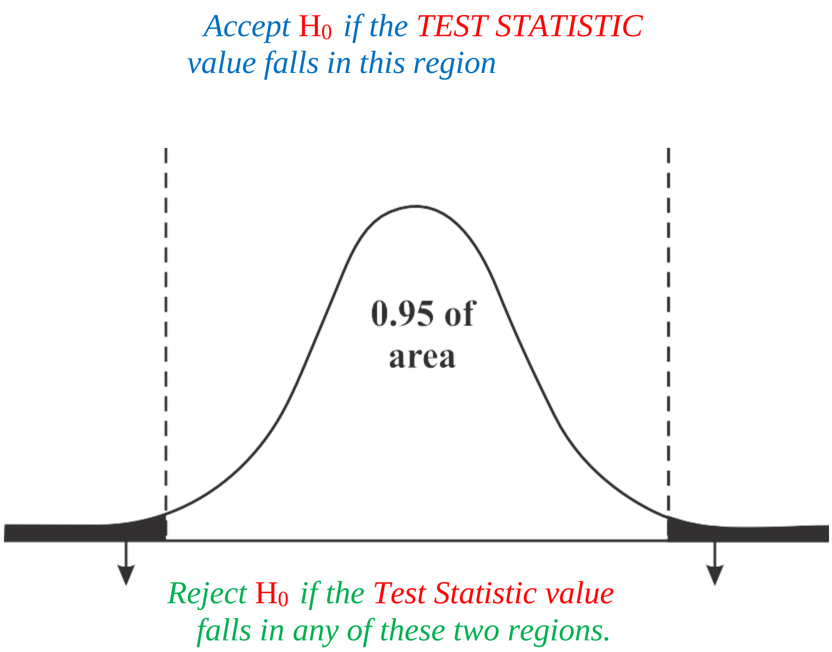
In testing a given hypothesis, the maximum probability with which we would be willing to take risk is called level of significance of the test. This probability often denoted by "α" is generally specified before samples are drawn.

The level of significance usually employed in testing of significance are

0.05 (or 5 %) and 0.01 (or 1 %).

For example, if 0.05 or 5 % level of significance is chosen in deriving a test of hypothesis, then there are about 5 chances in 100 that we would REJECT the hypothesis while it is ***TRUE***. (i.e.,) we could be wrong with probability 0.05 and we are about 95 % confident that we made the right decision.

The following diagram illustrates the region in which we could ACCEPT or REJECT the null-hypothesis at 5 % level of significance and the test is a Two-Tailed Test.



Critical Value:

The value of test statistic which separates the critical (or rejection) region and the acceptance region is called the critical value or significant value.

It depends upon i) the level of significance used and

ii) the alternative hypothesis (Two-tailed or Left-tailed or Right-tailed)

For large samples, the standard normal variate corresponding to the statistic t,

$$Z = \left| \frac{t - E(t)}{S.E.(t)} \right| \sim N(0,1) \quad \text{asymptotically as } n \to \infty$$

The value of Z, under H_0, is known as Test Statistic.

The critical value of the test statistic at the level of significance α,

for a two - tailed test is given by $Z_{\alpha/2}$ and

for a one tailed test (Right tailed or left tailed) by Z_α.

where Z_α is determined by equation $P(|Z| > Z_\alpha) = \alpha$.

3.7 One tailed and Two Tailed tests:

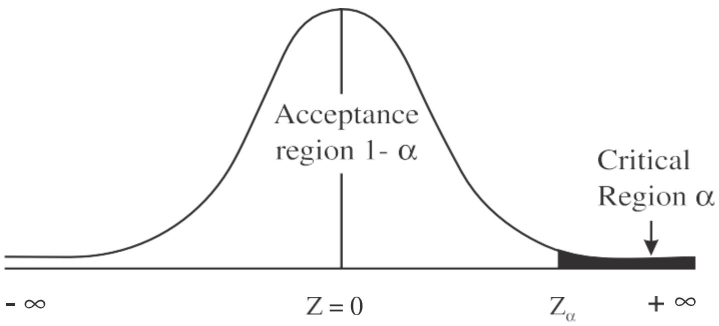
In any test, the critical region is represented by a portion of the area under the probability curve of the sampling distribution of the test statistic.

One tailed test: A test of any statistical hypothesis where the alternative hypothesis is one tailed (right tailed or left tailed) is called a one tailed test.

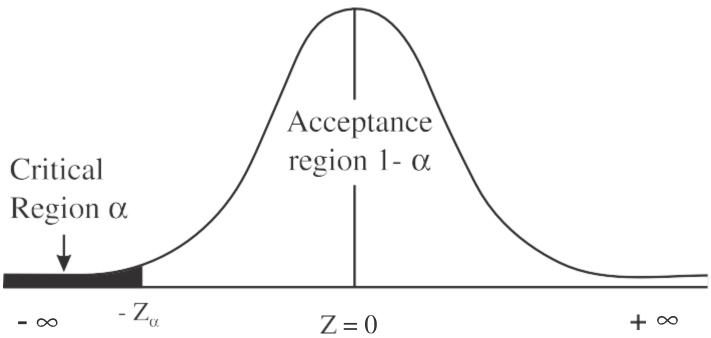
For example, for testing the mean of a population $H_0: \mu = \mu_0$, against the alternative hypothesis $H_1: \mu > \mu_0$ (right-tailed) or $H_1: \mu < \mu_0$ (left-tailed) is a one tailed test.

In the right- tailed test $H_1: \mu > \mu_0$, the critical region lies entirely in right tail of the sampling distribution of x, while for the left tailed test $H_1: \mu < \mu_0$ the critical region is entirely in the left of the distribution of x.

Right tailed test:



Left tailed test :



Two tailed test:

A test of hypothesis where the alternative hypothesis is two tailed such as,

$H_0 : \mu = \mu_0$ against the alternative hypothesis $H_1: \mu \neq \mu_0$ ($\mu > \mu_0$ and $\mu < \mu_0$)

is called a two tailed test.

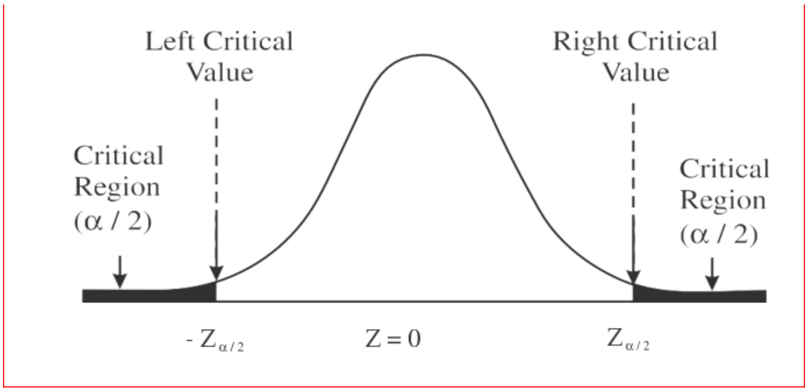
ie., the critical region is lying in both the tails of the probability curve of test of statistic.

$Z\alpha$ is the value so that the total area of the critical region on both tails is α.

$\therefore P(Z > Z_\alpha) = \frac{\alpha}{2}$. Area of each tail is $\frac{\alpha}{2}$.

$Z_{\alpha/2}$ is the value such that area to the right of $Z_{\alpha/2}$ and to the left of $-Z_{\alpha/2}$ is $\alpha/2$

as shown in the following diagram.



For example, suppose that there are two population brands of washing machines, are manufactured by standard process (with mean warranty period μ_1) and the other manufactured by some new technique (with mean warranty period μ_2).

If we want to test if the washing machines differ significantly then our null hypothesis is $H_0 : \mu_1 = \mu_2$ and alternative will be $H_1 : \mu_1 \neq \mu_2$ thus giving us a two tailed test.

However if we want to test whether the average warranty period produced by some new technique is more than those produced by standard process,

then we have $H_0 : \mu_1 = \mu_2$ and $H_1 : \mu_1 < \mu_2$ thus giving us a left-tailed test.

Similarly, for testing if the product of new process is inferior to that of standard process then we have, $H_0 : \mu_1 = \mu_2$ and $H_1 : \mu_1 > \mu_2$ thus giving us a right-tailed test.

Thus, the decision about applying a two-tailed test or a single –tailed (right or left) test will depend on the problem under study.

Zα: Critical values of Z

Level of significance (α)	Two-tailed	Left-tailed	Right-tailed
5% (0.05)	1.96	– 1.645	1.645
1% (0.01)	2.58	– 2.33	2.33

3.6 Large Sample Tests

General Test Procedure for Test of Hypothesis

(Common for Both Large Sample & Small Sample Tests)

The following steps constitute a general procedure, which can be followed for solving hypotheses testing problems based on both large and small samples.

Step 1 : Describe the population and its parameter(s). Frame the null hypothesis (H_0) and alternative hypothesis (H_1).

Step 2 : Describe the sample *i.e.*, data.

Step 3 : Specify the desired level of significance, α.

Step 4 : Specify the test statistic and its sampling distribution under H_0.

Step 5 : Calculate the value of the test statistic under H_0 for given sample.

Step 6 : Find the critical value(s) (table value(s)) from the statistical table generated from the sampling distribution of the test statistic under H_0 corresponding to α.

Step 7 : Decide on rejecting or not rejecting the null hypothesis based on the rejection rule which compares the calculated value(s) of the test statistic with the table value(s).

In practical problems, statisticians are supposed to make tentative calculations basedon sample observations. For example

(i) The average weight of school student is 35kg.

(ii) The coin is unbiased.

Now to reach such decisions it is essential to make certain assumptions (or guesses) about a population parameter. Such an assumption is known as statistical hypothesis, the validity of which is to be tested by analysing the sample. The procedure, which decides a certain hypothesis is true or false, is called the test of hypothesis (or test of significance).

Let us assume a certain value for a population mean. To test the validity of our assumption, we collect sample data and determine the difference between the hypothesized value and the actual value of the sample mean. Then, we judge whether the difference is significant or not. If the difference is smaller, there is a greater chance to accept our hypothesized value for the mean as correct. The larger the difference there is a greater chance to reject our hypothesized value for the mean as not correct.

Large samples (n > 30):

The tests of Hypothesis used for problems of large samples are different from those used in case of small samples as the assumptions used in both cases are different. The following assumptions are made for problems dealing with large samples:

- Almost all the sampling distributions follow normal asymptotically.
- The sample values are approximately close to the population values.

The following tests are the general tests used in large sample tests.

A. Test of Hypothesis for Population Mean ($H_0: \mu = \mu_0$)
  - **A.1 Population Variance is known (σ^2 known)**
  - **A.2 Population Variance is unknown (σ^2 unknown)**

B. Test of Hypothesis for Equality of Two Population Means. ($H_0: \mu_1 = \mu_2$)
  - **B.1 Population Variances are known (σ_1^2 & σ_2^2 known)**
  - **B.2 Population Variances are unknown (σ_1^2 & σ_2^2 unknown)**

C. Test of Hypothesis for Population Proportion ($H_0: P = P_0$)

D. Test of Hypothesis for Equality of Two Population Proportions ($H_0: P_1 = P_2$)

A.1. TEST OF HYPOTHESES FOR POPULATION MEAN (Population variance is known)

Procedure:

Step 1 : Let μ and σ^2 be respectively the mean and the variance of the population under study, where σ^2 is known. If μ_0 is an admissible value of μ, then frame the null hypothesis as $H_0: \mu = \mu_0$ and choose the suitable alternative hypothesis from

(i) $H_1: \mu \neq \mu_0$ (ii) $H_1: \mu > \mu_0$ (iii) $H_1: \mu < \mu_0$

Step 2 : Let $(X_1, X_2, \ldots, X_n)$ be a random sample of n observations drawn from the population, where n is large ($n \geq 30$).

Step 3 : Let the level of significance be α.

Step 4 : Consider the test statistic $Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$ under H_0. Here, $\bar{X}$ represents the sample mean, which is defined in Note 2. The approximate sampling distribution of the test statistic under H_0 is the $N(0,1)$ distribution.

Step 5 : Calculate the value of Z for the given sample $(x_1, x_2, \ldots, x_n)$ as

$$z_0 = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$$

Step 6 : Find the critical value, z_e, corresponding to α and H_1 from the following table

Alternative Hypothesis (H_1)	$\mu \neq \mu_0$	$\mu > \mu_0$	$\mu < \mu_0$
Critical Value (z_e)	$z_{\alpha/2}$	z_α	$-z_\alpha$

Step 7 : Decide on H_0 choosing the suitable rejection rule from the following table corresponding to H_1.

Alternative Hypothesis (H_1)	$\mu \neq \mu_0$	$\mu > \mu_0$	$\mu < \mu_0$
Rejection Rule	$\lvert z_0 \rvert \geq z_{\alpha/2}$	$z_0 > z_\alpha$	$z_0 < -z_\alpha$

Zα: Critical values of Z

Level of significance (α)	Two-tailed	Left-tailed	Right-tailed
5% (0.05)	1.96	– 1.645	1.645
1% (0.01)	2.58	– 2.33	2.33

Example: 1 (Two-sided Test)

A company producing LED bulbs finds that mean life span of the population of its bulbs is 2000 hours with a standard derivation of 150 hours. A sample of 100 bulbs randomly chosen is found to have the mean life span of 1950 hours. Test, at 5% level of significance, whether the mean life span of the bulbs is significantly different from 2000 hours.

Solution:

Step 1 : Let μ and σ represent respectively the mean and standard deviation of the probability distribution of the life span of the bulbs. It is given that $\sigma = 150$ hours. The null and alternative hypotheses are

Null hypothesis: H_0: $\mu = 2000$

i.e., the mean life span of the bulbs is not significantly different from 2000 hours.

Alternative hypothesis: $H_1 : \mu \neq 2000$

i.e., the mean life span of the bulbs is significantly different from 2000 hours.

It is a two-sided alternative hypothesis.

Step 2 : **Data**

The given sample information are

Sample size $(n) = 100$, Sample mean $(\bar{x}) = 1950$ hours

Step 3 : **Level of significance**

$\alpha = 5\%$

Step 4 : **Test statistic**

The test statistic is $Z = \dfrac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$, under H_0

Under the null hypothesis H_0, Z follows the $N(0,1)$ distribution.

Step 5 : **Calculation of Test Statistic**

The value of Z under H_0 is calculated from

$$z_0 = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$$

as

$$z_0 = \frac{1950 - 2000}{150/\sqrt{100}}$$

$$= -3.33$$

Thus; $|z_0| = 3.33$

Step 6 : **Critical value**

Since H_1 is a two-sided alternative, the critical value at $\alpha = 0.05$ is $z_e = z_{0.025} = 1.96$. (*see* Table 1.6).

Step 7 : **Decision**

Since H_1 is a two-sided alternative, elements of the critical region are determined by the rejection rule $|z_0| \geq z_e$. Thus, it is a two-tailed test. For the given sample information, the rejection rule holds *i.e.*, $|z_0| = 3.33 > z_e = 1.96$. Hence, H_0 is rejected in favour of H_1: $\mu \neq 2000$. Thus, the mean life span of the LED bulbs is significantly different from 2000 hours.

Example: 2 (Right-sided Test)

The mean breaking strength of cables supplied by a manufacturer is 1900 n/m^2 with a standard deviation of 120 n/m^2. The manufacturer introduced a new technique in the manufacturing process and claimed that the breaking strength of the cables has increased. In order to test the claim, a sample of 60 cables is tested. It is found that the mean breaking strength of the sampled cables is 1960 n/m^2. Can we support the claim at 1% level of significance?

Solution:

Step 1 : Let μ and σ represent respectively the mean and standard deviation of the probability distribution of the breaking strength of the cables. It is given that $\sigma = 120$ n/m^2. The null and alternative hypotheses are

Null hypothesis $H_0: \mu = 1900$

i.e., the mean breaking strength of the cables is not significantly different from $1900 n/m^2$.

Alternative hypothesis: $H_1: \mu > 1900$

i.e., the mean breaking strength of the cables is significantly more than $1900 n/m^2$.

It may be noted that it is a one-sided (right) alternative hypothesis.

Step 2 : **Data**

The given sample information are

Sample size (n) = 60. Hence, it is a large sample.

Sample mean ($\bar{x}$)= 1960

Step 3 : **Level of significance**

$\alpha = 1\%$

Step 4 : **Test statistic**

The test statistic is $Z = \dfrac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$, under H_0

Since n is large, under the null hypothesis, the sampling distribution of Z is the $N(0,1)$ distribution.

Step 5 : **Calculation of test statistic**

The value of Z under H_0 is calculated from $z_0 = \dfrac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$

$$z_0 = \frac{1960 - 1900}{120 / \sqrt{60}}$$

Thus, $z_0 = 3.87$

Step 6 : **Critical value**

Since H_1 is a one-sided (right) alternative hypothesis, the critical value at $\alpha = 0.01$ level of significance is $z_e = z_{0.01} = 2.33$ (*see* Table 1.6)

Step 7 : **Decision**

Since H_1 is a one-sided (right) alternative, elements of the critical region are determined by the rejection rule $z_0 > z_e$. Thus, it is a right-tailed test. For the given sample information, the observed value $z_0 = 3.87$ is greater than the critical value $z_e = 2.33$. Hence, the null hypothesis H_0 is rejected. Therefore, the mean breaking strength of the cables is significantly more than 1900 n/m^2.

Thus, the manufacturer's claim that the breaking strength of cables has increased by the new technique is found valid.

A.2. TEST OF HYPOTHESES FOR POPULATION MEAN (POPULATION VARIANCE IS UNKNOWN)

Procedure:

Step1 : Let μ and σ^2 be respectively the mean and the variance of the population under study, where σ^2 is unknown. If μ_0 is an admissible value of μ, then frame the null hypothesis as H_0: $\mu = \mu_0$ and choose the suitable alternative hypothesis from

(i) H_1: $\mu \neq \mu_0$ (ii) H_1: $\mu > \mu_0$ (iii) H_1: $\mu < \mu_0$

Step 2 : Let $(X_1, X_2, \ldots, X_n)$ be a random sample of n observations drawn from the population, where n is large ($n \geq 30$).

Step 3 : Specify the level of significance, α.

Step 4 : Consider the test statistic $Z = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}}$ under H_0, where $\bar{X}$ and S are the sample mean and sample standard deviation respectively. It may be noted that the above test statistic is obtained from Z considered in the test described in Section 1.9 by substituting S for σ.

The approximate sampling distribution of the test statistic under H_0 is the $N(0,1)$ distribution.

YOU WILL KNOW

It is important to note that the exact sampling distribution of Z is the *Student's 't'* distribution with $(n - 1)$ degrees of freedom, when n is small ($n < 30$). This hypotheses testing problem, when n is small, is discussed, in detail, in Chapter 2. When n is large, the *Student's 't'* distribution converges to the $N(0,1)$ distribution.

Step 5 : Calculate the value of Z for the given sample $(x_1, x_2, \ldots, x_n)$ as $z_0 = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$. Here, $\bar{x}$ and s are respectively the values of $\bar{X}$ and S calculated for the given sample.

Step 6 : Find the critical value, z_e, corresponding to α and H_1 from the following table

Alternative Hypothesis (H_1)	$\mu \neq \mu_0$	$\mu > \mu_0$	$\mu < \mu_0$
Critical Value (z_e)	$z_{\alpha/2}$	z_α	$-z_\alpha$

Step 7 : Decide on H_0 choosing the suitable rejection rule from the following table corresponding to H_1.

Alternative Hypothesis (H_1)	$\mu \neq \mu_0$	$\mu > \mu_0$	$\mu < \mu_0$
Rejection Rule	$\lvert z_0 \rvert \geq z_{\alpha/2}$	$z_0 > z_\alpha$	$z_0 < -z_\alpha$

Example: 3 (Right-sided Test)

A motor vehicle manufacturing company desires to introduce a new model motor vehicle. The company claims that the mean fuel consumption of its new model vehicle is lower than that of the existing model of the motor vehicle, which is 27 kms/litre. A sample of 100 vehicles of the new model vehicle is selected randomly and their fuel consumptions are observed. It is found that the mean fuel consumption of the 100 new model motor vehicles is 30 kms/litre with a standard deviation of 3 kms/litre. Test the claim of the company at 5% level of significance.

Solution:

Step 1 : Let the fuel consumption of the new model motor vehicle be assumed to be distributed according to a distribution with mean and standard deviation respectively μ and σ. The null and alternative hypotheses are

Null hypothesis H_0: $\mu = 27$

i.e., the average fuel consumption of the company's new model motor vehicle is not significantly different from that of the existing model.

Alternative hypothesis H_1: $\mu > 27$

i.e., the average fuel consumption of the company's new model motor vehicle is significantly lower than that of the existing model. In other words, the number of kms by the new model motor vehicle is significantly more than that of the existing model motor vehicle.

Step 2 : **Data:**

The given sample information are

Size of the sample (n) = 100. Hence, it is a large sample.

Sample mean ($\bar{x}$) = 30

Sample standard deviation(s) = 3

Step 3 : **Level of significance**

$\alpha = 5\%$

Step 4 : **Test statistic**

The test statistic under H_0 is

$$Z = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}.$$

Since n is large, the sampling distribution of Z under H_0 is the $N(0,1)$ distribution.

Step 5 : **Calculation of Test Statistic**

The value of Z for the given sample information is calculated from

$$z_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \text{ as}$$

$$z_0 = \frac{30 - 27}{3/\sqrt{100}}$$

Thus, $z_0 = 10$.

Step 6 : **Critical Value**

Since H_1 is a one-sided (right) alternative hypothesis, the critical value at $\alpha = 0.05$ is $z_e = z_{0.05} = 1.645$.

Step 7 : **Decision**

Since H_1 is a one-sided (right) alternative, elements of the critical region are defined by the rejection rule $z_0 > z_e = z_{0.05}$. Thus, it is a right-tailed test. Since, for the given sample information, $z_0 = 10 > z_e = 1.645$, H_0 is rejected.

B.1. TEST OF HYPOTHESES FOR EQUALITY OF MEANS OF TWO POPULATIONS *(Population variances are known)*

Procedure:

Step-1 : Let μ_X and σ_X^2 be respectively the mean and the variance of Population -1. Also, let μ_Y and σ_Y^2 be respectively the mean and the variance of Population -2 under study. Here σ_X^2 and σ_Y^2 are known admissible values.

Frame the null hypothesis as $H_0: \mu_X = \mu_Y$ and choose the suitable alternative hypothesis from

(i) $H_1: \mu_X \neq \mu_Y$ (ii) $H_1: \mu_X > \mu_Y$ (iii) $H_1: \mu_X < \mu_Y$

Step 2 : Let $(X_1, X_2, \ldots, X_m)$ be a random sample of m observations drawn from Population-1 and $(Y_1, Y_2, \ldots, Y_n)$ be a random sample of n observations drawn from Population-2, where m and n are large(*i.e.*, $m \geq 30$ and $n \geq 30$). Further, these two samples are assumed to be independent.

Step 3 : Specify the level of significance, α.

Step 4 : Consider the test statistic $Z = \frac{(\overline{X}-\overline{Y})-(\mu_X-\mu_Y)}{\sqrt{\frac{\sigma_X^2}{m}+\frac{\sigma_Y^2}{n}}}$ under H_0, where $\overline{X}$ and $\overline{Y}$ are respectively the means of the two samples described in Step-2.

The approximate sampling distribution of the test statistic $Z = \frac{(\overline{X}-\overline{Y})}{\sqrt{\frac{\sigma_X^2}{m}+\frac{\sigma_Y^2}{n}}}$ under H_0 (*i.e.*, $\mu_X = \mu_Y$) is the $N(0,1)$ distribution.

It may be noted that the test statistic, when $\sigma_X^2 = \sigma_Y^2 = \sigma^2$, is $Z = \frac{(\overline{X}-\overline{Y})}{\sigma\sqrt{\frac{1}{m}+\frac{1}{n}}}$.

Step 5 : Calculate the value of Z for the given samples $(x_1, x_2, \ldots, x_m)$ and $(y_1, y_2, \ldots, y_n)$ as $z_0 = \frac{(\overline{x}-\overline{y})}{\sqrt{\frac{\sigma_X^2}{m}+\frac{\sigma_Y^2}{n}}}$.

Here, $\overline{x}$ and $\overline{y}$ are respectively the values of $\overline{X}$ and $\overline{Y}$ for the given samples.

Step 6 : Find the critical value, z_e, corresponding to α and H_1 from the following table

Alternative Hypothesis (H_1)	$\mu_X \neq \mu_Y$	$\mu_X > \mu_Y$	$\mu_X < \mu_Y$
Critical Value (z_e)	$z_{\alpha/2}$	z_α	$-z_\alpha$

Step 7 : Make decision on H_0 choosing the suitable rejection rule from the following table corresponding to H_1.

Alternative Hypothesis (H_1)	$\mu_X \neq \mu_Y$	$\mu_X > \mu_Y$	$\mu_X < \mu_Y$
Rejection Rule	$\lvert z_0 \rvert \geq z_{\alpha/2}$	$z_0 > z_\alpha$	$z_0 < -z_\alpha$

Example:

Performance of students of X Standard in a national level talent search examination was studied. The scores secured by randomly selected students from two districts, *viz.*, D_1 and D_2 of a State were analyzed. The number of students randomly selected from D_1 and D_2 are respectively 500 and 800. Average scores secured by the students selected from D_1 and D_2 are respectively 58 and 57. Can the samples be regarded as drawn from the identical populations having common standard deviation 2? Test at 5% level of significance.

Solution:

Step 1 : Let μ_X and μ_Y be respectively the mean scores secured in the national level talent search examination by all the students from the districts D_1 and D_2 considered for the study. It is given that the populations of the scores of the students of these districts have the common standard deviation $\sigma = 2$. The null and alternative hypotheses are

Null hypothesis: H_0: $\mu_X = \mu_Y$

i.e., average scores secured by the students from the study districts are not significantly different.

Alternative hypothesis: H_1: $\mu_X \neq \mu_Y$

i.e., average scores secured by the students from the study districts are significantly different. It is a two-sided alternative.

Step 2 : **Data**

The given sample information are

Size of the Sample-1 (m) = 500

Size of the Sample-2 (n) = 800. Hence, both the samples are large.

Mean of Sample-1 ($\bar{x}$) = 58

Mean of Sample-2 ($\bar{y}$) = 57

Step 3 : **Level of significance**

$\alpha = 5\%$

Step 4 : **Test statistic**

The test statistic under the null hypothesis H_0 is

$$Z = \frac{\bar{X} - \bar{Y}}{\sigma\sqrt{\frac{1}{m} + \frac{1}{n}}}.$$

Since both m and n are large, the sampling distribution of Z under H_0 is the $N(0, 1)$ distribution.

Step 5 : **Calculation of Test Statistic**

The value of Z is calculated for the given sample information from

$$z_0 = \frac{\bar{x} - \bar{y}}{\sigma\sqrt{\frac{1}{m} + \frac{1}{n}}} \quad \text{as}$$

$$z_0 = \frac{58 - 57}{2\sqrt{\frac{1}{500} + \frac{1}{800}}}$$

$$z_0 = 8.77$$

Step-6 : **Critical value**

Since H_1 is a two-sided alternative hypothesis, the critical value at $\alpha = 0.05$ is $z_e = z_{0.025} = 1.96$.

Step-7 : **Decision**

Since H_1 is a two-sided alternative, elements of the critical region are defined by the rejection rule $|z_0| \geq z_e = z_{0.025}$. For the given sample information, $|z_0| = 8.77 > z_e = 1.96$. It indicates that the given sample contains sufficient evidence to reject H_0. Thus, it may be decided that H_0 is rejected. Therefore, the average performance of the students in the districts D_1 and D_2 in the national level talent search examination are significantly different. Thus the given samples are not drawn from identical populations.

B.2. TEST OF HYPOTHESES FOR EQUALITY OF MEANS OF TWO POPULATIONS (*POPULATION VARIANCES ARE UNKNOWN*)

Procedure:

Step-1 : Let μ_X and σ_X^2 be respectively the mean and the variance of Population -1. Also, let μ_Y and σ_Y^2 be respectively the mean and the variance of Population -2 under study. Here σ_X^2 and σ_Y^2 are assumed to be unknown.

Frame the null hypothesis as H_0: $\mu_X = \mu_Y$ and choose the suitable alternative hypothesis from

(i) H_1: $\mu_X \neq \mu_Y$ (ii) H_1: $\mu_X > \mu_Y$ (iii) H_1: $\mu_X < \mu_Y$

Step 2 : Let $(X_1, X_2, \ldots, X_m)$ be a random sample of m observations drawn from Population-1 and $(Y_1, Y_2, \ldots, Y_n)$ be a random sample of n observations drawn from Population-2, where m and n are large ($m \geq 30$ and $n \geq 30$). Here, these two samples are assumed to be independent.

Step 3 : Specify the level of significance, α.

Step 4 : Consider the test statistic

$$Z = \frac{(\overline{X} - \overline{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{S_X^2}{m} + \frac{S_Y^2}{n}}} \text{ under } H_0 \ (i.e., \mu_X = \mu_Y).$$

i.e., the above test statistic is obtained from Z considered in the test described in Section 1.11 by substituting S_X^2 and S_Y^2 respectively for σ_X^2 and σ_Y^2

The approximate sampling distribution of the test statistic $Z = \frac{\overline{X} - \overline{Y}}{\sqrt{\frac{S_X^2}{m} + \frac{S_Y^2}{n}}}$ under H_0 is the $N(0,1)$ distribution.

Step 5 : Calculate the value of Z for the given samples $(x_1, x_2 \ldots, x_m)$ and $(y_1, y_2, \ldots, y_n)$ as

$$z_0 = \frac{\overline{x} - \overline{y}}{\sqrt{\frac{s_X^2}{m} + \frac{s_Y^2}{n}}}$$

Here $\overline{x}$ and $\overline{y}$ are respectively the values of $\overline{X}$ and $\overline{Y}$ for the given samples. Also, s_X^2 *and* s_Y^2 are respectively the values of S_X^2 and S_Y^2 for the given samples.

Step 6 : Find the critical value, z_e, corresponding to α and H_1 from the following table

Alternative Hypothesis (H_1)	$\mu_X \neq \mu_Y$	$\mu_X > \mu_Y$	$\mu_X < \mu_Y$
Critical Value (z_e)	$z_{\alpha/2}$	z_α	$-z_\alpha$

Step 7 : Make decision on H_0 choosing the suitable rejection rule from the following table corresponding to H_1.

Alternative Hypothesis (H_1)	$\mu_X \neq \mu_Y$	$\mu_X > \mu_Y$	$\mu_X < \mu_Y$
Rejection Rule	$\lvert z_0 \rvert \geq z_{\alpha/2}$	$z_0 > z_\alpha$	$z_0 < -z_\alpha$

Example:

A Model Examination was conducted to XII Standard students in the subject of Statistics. A District Educational Officer wanted to analyze the Gender-wise performance of the students using the marks secured by randomly selected boys and girls. Sample measures were calculated and the details are presented below:

Gender	Sample Size	Sample Mean	Sample Standard deviation
Boys	100	50	4
Girls	150	51	5

Test, at 5% level of significance, whether performance of the students differ significantly with respect to their gender.

Solution:

Step 1 : Let μ_X and μ_Y denote respectively the average marks secured by boys and girls in the Model Examination conducted to the XII Standard students in the subject of Statistics. Then, the null and the alternative hypotheses are

Null hypothesis: $H_0: \mu_X = \mu_Y$

i.e., there is no significant difference in the performance of the students with respect to their gender.

Alternative hypothesis: $H_1: \mu_X \neq \mu_Y$

i.e., performance of the students differ significantly with the respect to the gender. It is a two-sided alternative hypothesis.

Step 2 : **Data**

The given sample information are

Gender of the Students	Sample Size	Sample Mean	Sample Standard Deviation
Boys	$m = 100$	$\bar{x} = 50$	$s_X = 4$
Girls	$n = 150$	$\bar{y} = 51$	$s_Y = 5$

Since $m \geq 30$ and $n \geq 30$, both the samples are large.

Step 3 : **Level of significance**

$\alpha = 5\%$

Step 4 : **Test statistic**

The test statistic under H_0 is

$$Z = \frac{\overline{X} - \overline{Y}}{\sqrt{\frac{S_X^2}{m} + \frac{S_Y^2}{n}}}.$$

The sampling distribution of Z under H_0 is the $N(0,1)$ distribution.

Step 5 : **Calculation of the Test Statistic**

The value of Z is calculated for the given sample informations from

$$z_0 = \frac{\overline{x} - \overline{y}}{\sqrt{\frac{s_X^2}{m} + \frac{s_Y^2}{n}}} \quad \text{as}$$

$$z_0 = \frac{50 - 51}{\sqrt{\frac{4^2}{100} + \frac{5^2}{150}}}$$

Thus, $z_0 = -1.75$

Step 6 : **Critical value**

Since H_1 is a two-sided alternative, the critical value at 5% level of significance is $z_e = z_{0.025} = 1.96$.

Step 7 : **Decision**

Since H_1 is a two-sided alternative, elements of the critical region are determined by the rejection rule $|z_0| \geq |z_e|$. Thus it is a two-tailed test. But, $|z_0| = 1.75$ is less than the critical value $z_e = 1.96$. Hence, it may inferred as the given sample information does not provide sufficient evidence to reject H_0. Therefore, it may be decided that there is no sufficient evidence in the given sample to conclude that performance of boys and girls in the Model Examination conducted in the subject of Statistics differ significantly.

C. TEST OF HYPOTHESES FOR POPULATION PROPORTION

Procedure:

Step 1 : Let P denote the proportion of the population possessing the qualitative characteristic (attribute) under study. If p_0 is an admissible value of P, then frame the null hypothesis as $H_0: P = p_0$ and choose the suitable alternative hypothesis from

(i) $H_1: P \neq p_0$ (ii) $H_1: P > p_0$ (iii) $H_1: P < p_0$

Step 2 : Let p be proportion of the sample observations possessing the attribute, where n is large, $np > 5$ and $n(1 - p) > 5$.

Step 3 : Specify the level of significance, α.

Step 4 : Consider the test statistic $Z = \frac{p - P}{\sqrt{\frac{PQ}{n}}}$ under H_0. Here, $Q = 1 - P$.

The approximate sampling distribution of the test statistic under H_0 is the $N(0,1)$ distribution.

Step 5 : Calculate the value of Z under H_0 for the given data as $z_0 = \frac{p - p_0}{\sqrt{\frac{p_0 q_0}{n}}}$, $q_0 = 1 - p_0$.

Step 6 : Choose the critical value, z_e, corresponding to α and H_1 from the following table

Alternative Hypothesis (H_1)	$P \neq p_0$	$P > p_0$	$P < p_0$
Critical Value (z_e)	$z_{\alpha/2}$	z_α	$-z_\alpha$

Step 7 : Make decision on H_0 choosing the suitable rejection rule from the following table corresponding to H_1.

Alternative Hypothesis (H_1)	$P \neq p_0$	$P > p_0$	$P < p_0$
Rejection Rule	$\lvert z_0 \rvert \geq z_{\alpha/2}$	$z_0 > z_\alpha$	$z_0 < -z_\alpha$

Example:

A survey was conducted among the citizens of a city to study their preference towards consumption of tea and coffee. Among 1000 randomly selected persons, it is found that 560 are tea-drinkers and the remaining are coffee-drinkers. Can we conclude at 1% level of significance from this information that both tea and coffee are equally preferred among the citizens in the city?

Solution:

Step 1 : Let P denote the proportion of people in the city who preferred to consume tea. Then, the null and the alternative hypotheses are

Null hypothesis: $H_0 : P = 0.5$

i.e., it is significant that both tea and coffee are preferred equally in the city.

Alternative hypothesis: $H_1 : P \neq 0.5$

i.e., preference of tea and coffee are not significantly equal. It is a two-sided alternative hypothesis.

Step 2 : **Data**

The given sample information are

Sample size (n) = 1000. Hence, it is a large sample.

No. of tea-drinkers = 560

Sample proportion (p) = $\frac{560}{1000} = 0.56$

Step 3 : **Level of significance**

$\alpha = 1\%$

Step 4 : Test statistic

Since n is large, $np = 560 > 5$ and $n(1 - p) = 440 > 5$, the test statistic under the null hypothesis, is $Z = \frac{p-P}{\sqrt{\frac{PQ}{n}}}$.

Its sampling distribution under H_0 is the $N(0,1)$ distribution.

Step 5 : Calculation of Test Statistic

The value of Z can be calculated for the sample information from

$z_0 = \frac{p-p_0}{\sqrt{\frac{p_0 q_0}{n}}}$ as

$$z_0 = \frac{0.56-0.50}{\sqrt{\frac{0.5\times0.5}{1000}}}$$

Thus, $z_0 = 3.79$

Step 6 : Critical value

Since H_1 is a two-sided alternative hypothesis, the critical value at 1% level of significance is $z_{\alpha/2} = z_{0.005} = 2.58$.

Step 7 : Decision

Since H_1 is a two-sided alternative, elements of the critical region are determined by the rejection rule $|z_0| \geq z_e$. Thus it is a two-tailed test. Since $|z_0| = 3.79 > z_e = 2.58$, reject H_0 at 1% level of significance. Therefore, there is significant evidence to conclude that the preference of tea and coffee are different.

D. TEST OF HYPOTHESES FOR EQUALITY OF PROPORTIONS OF TWO POPULATIONS

Procedure:

Step 1 : Let P_X and P_Y denote respectively the proportions of Population-1 and Population-2 possessing the qualitative characteristic (attribute) under study. Frame the null hypothesis as H_0: $P_X=P_Y$ and choose the suitable alternative hypothesis from

(i) H_1: $P_X \neq P_Y$ (ii) H_1: $P_X>P_Y$ (iii) H_1: $P_X<P_Y$

Step 2 : Let p_X and p_Y **denote respectively the proportions of the samples of sizes m and n** drawn from Population-1 and Population-2 possessing the attribute, where m and n are large (*i.e.*, $m \geq 30$ and $n \geq 30$). Also, $mp_X > 5$, $m(1-p_X) > 5$, $np_Y > 5$ and $n(1-p_Y) > 5$. Here, these two samples are assumed to be independent.

Step 3 : Specify the level of significance, α.

Step 4 : Consider the test statistic $Z = \frac{(p_X - p_Y) - (P_X - P_Y)}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{m} + \frac{1}{n}\right)}}$ under H_0. Here, $\hat{p} = \frac{mp_X + np_Y}{m+n}$, $\hat{q} = 1 - \hat{p}$. The approximate sampling distribution of the test statistic under H_0 is the $N(0,1)$ distribution.

Step 5 : Calculate the value of Z for the given data as $z_0 = \frac{p_X - p_Y}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{m} + \frac{1}{n}\right)}}$.

Step 6 : Choose the critical value, z_e, corresponding to α and H_1 from the following table

Alternative Hypothesis (H_1)	$P_X \neq P_Y$	$P_X > P_Y$	$P_X < P_Y$
Critical Value (z_e)	$z_{\alpha/2}$	z_α	$-z_\alpha$

Step 7 : Decide on H_0 choosing the suitable rejection rule from the following table corresponding to H_1.

Alternative Hypothesis (H_1)	$P_X \neq P_Y$	$P_X > P_Y$	$P_X < P_Y$
Rejection Rule	$\lvert z_0 \rvert \geq z_{\alpha/2}$	$z_0 > z_\alpha$	$z_0 < -z_\alpha$

Example:

A study was conducted to investigate the interest of people living in cities towards self-employment. Among randomly selected 500 persons from City-1, 400 persons were found to be self-employed. From City-2, 800 persons were selected randomly and among them 600 persons are self-employed. Do the data indicate that the two cities are significantly different with respect to prevalence of self-employment among the persons? Choose the level of significance as $\alpha = 0.05$.

Solution:

Step1 : Let P_X and P_Y be respectively the proportions of self-employed people in City-1 and City-2. Then, the null and alternative hypotheses are

Null hypothesis: $H_0 : P_X = P_Y$

i.e., there is no significant difference between the proportions of self-employed people in City-1 and City-2.

Alternative hypothesis: $H_1 : P_X \neq P_Y$

i.e., difference between the proportions of self-employed people in City-1 and City-2 is significant. It is a two-sided alternative hypothesis.

Step 2 : Data

The given sample information are

City	Sample Size	Sample Proportion
City-1	$m = 500$	$p_X = \frac{400}{500} = 0.80$
City-2	$n = 800$	$p_Y = \frac{600}{800} = 0.75$

Here, $m \geq 30$, $n \geq 30$, $mp_X = 400 > 5$, $m(1-p_X) = 100 > 5$, $np_Y = 600 > 5$ and $n(1-p_Y) = 200 > 5$.

Step 3 : Level of significance

$\alpha = 5\%$

Step 4 : Test statistic

The test statistic under the null hypothesis is

$$Z = \frac{p_X - p_Y}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{m}+\frac{1}{n}\right)}} \quad \text{where } \hat{p} = \frac{mp_X + np_Y}{m+n} \text{ and } \hat{q} = 1-\hat{p}$$

The sampling distribution of Z under H_0 is the $N(0,1)$ distribution.

Step 5 : Calculation of Test Statistic

The value of Z for given sample information is calculated from

$$z_0 = \frac{p_X - p_Y}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{m}+\frac{1}{n}\right)}}.$$

$$\text{Now, } \hat{p} = \frac{400+600}{500+800} = \frac{1000}{1300} = 0.77 \quad \text{and } \hat{q} = 0.23$$

$$\text{Thus, } z_0 = \frac{0.80-0.75}{\sqrt{(0.77)(0.23)\left(\frac{1}{500}+\frac{1}{800}\right)}}$$

$z_0 = 2.0764$

Step 6 : Critical value

Since H_1 is a two-sided alternative hypothesis, the critical value at 5% level of significance is $z_e = 1.96$.

Step 7 : Decision

Since H_0 is a two-sided alternative, elements of the critical region are determined by the rejection rule $|z_0| > z_e$. Thus, it is a two-tailed test. For the given sample information, $z_e = 2.0764 > z_e = 1.96$. Hence, H_0 is rejected. We can conclude that the difference between the proportions of self-employed people in City-1 and City-2 is significant.