UNIT-V : PROBABILITY

5.1 Concept of Probability

The word probability and chance are quite familiar to everyone. Many a time we come across statements like "There is a bright **chance** for Indian cricket team to win the current series this time".

"It is **possible** that a particular school students may get state ranks in forthcoming public examination".

"**Probably** it may rain today".

The word chance, possible, probably, likely etc. convey some sense of uncertainty about the occurrence of some events. Our entire world is filled with uncertainty. We make decisions affected by uncertainty virtually every day.

In order to think about and measure uncertainty, we turn to a branch of mathematics called probability.

5.2 Basic Concepts, Events, Equally Likely & Mutually Exclusive Events

EXPERIMENT

Deterministic Experiment **Random Experiment**

(Genetic determination)

(Hitting the target)

Definition 1

An **experiment** is defined as a process for which its result is well defined.

Definition 2

Deterministic experiment is an experiment whose outcomes can be predicted with certain, under ideal conditions.

Definition 3

A random experiment (or non-deterministic) is an experiment

(i) whose all possible outcomes are known in advance,

(ii) whose each outcome is not possible to predict in advance, and

(iii) can be repeated under identical conditions.

A die is 'rolled', a fair coin is 'tossed' are examples for random experiments.

Definition 4

A **simple event** (or elementary event or sample point) is the most basic possible outcome of a random experiment and it cannot be decomposed further.

Illustration:

(1) (i) If a die is rolled, then the sample space $S = \{1, 2, 3, 4, 5, 6\}$

(ii) A coin is tossed, then the sample space $S = \{H, T\}$

(2) (i) Suppose we toss a coin until a head is obtained. One cannot say in advance how many tosses will be required, and so the sample space.
$S = \{H, TH, TTH, TTTH, ...\}$ is an infinite set.

(ii) The sample space associated with the number of passengers waiting to buy train tickets in counters is $S = \{0,1,2,...\}$.

(3) (i) If the experiment consists of choosing a number randomly between 0 and 1, then the sample space is $S = \{x: 0 < x < 1\}$.

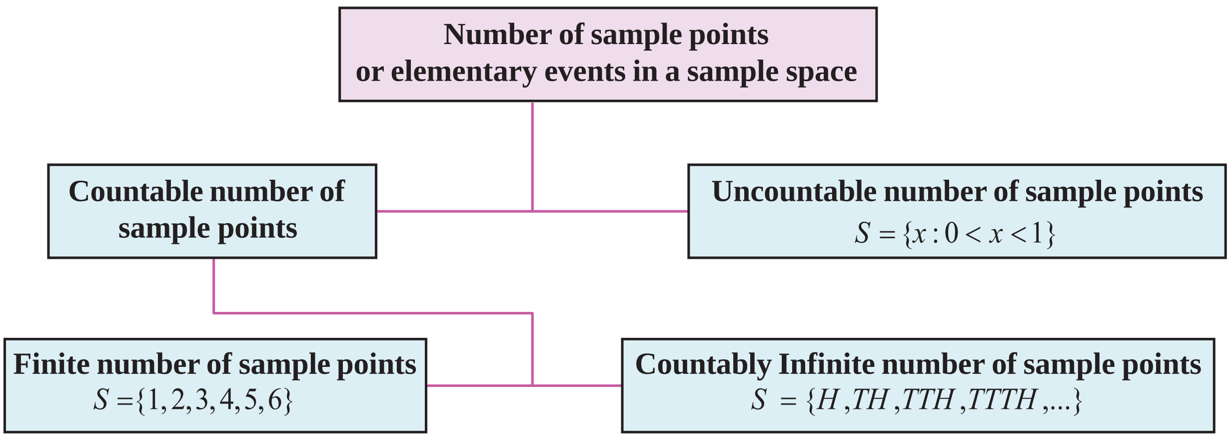
(ii) The sample space for the life length (t in hours) of a tube light is
$S = \{t: 0 < t < 1000\}$.

Definition 5

A sample space is the set of all possible outcomes of a random experiment. Each point in sample space is an elementary event.

From (2) and (3), one need to distinguish between two types of infinite sets, where one type is significantly 'larger' than the other. In particular, S in (2) is called **countably infinite**, while the S in (3) is called **uncountably infinite**. The fact that one can list the elements of a countably infinite set means that the set can be put in one-to-one correspondence with natural numbers □ . On the other hand, you cannot list the elements in uncountable set.

From the above example, one can understand that the sample space may consist of countable or uncountable number of elementary events.



Finite sample space

We restrict our sample spaces that have at most a finite number of points.

Types of events

Let us now define some of the important types of events, which are used frequently

- Sure event or certain event
- Impossible event
- Complementary event
- Mutually exclusive events
- Mutually inclusive event
- Exhaustive events

- Independent events
- Equally likely events

Definition 6

When the sample space is finite, any subset of the sample space is an **event**. That is, all elements of the power set $\mathscr{P}(S)$ of the sample space are defined as **events**. An event is a collection of sample points or elementary events.

The sample space S is called **sure event or certain event**. The null set $\varnothing$ in S is called an **impossible event**.

Definition 7

For every event A, there corresponds another event $\bar{A}$ is called the **complementary event** to A. It is also called the event 'not A'.

Illustration

Suppose a sample space S is given by $S = \{1,2,3,4\}$.

Let the set of all possible subsets of S (the power set of S) be $\mathscr{P}(S)$.

$$\mathscr{P}(S) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$$

(i) $\varnothing$ is an impossible event.

(ii) $\{1\},\{2\},\{3\},\{4\}$ are the simple events or elementary events.

(iii) $\{1, 2, 3, 4\}$ is a sure event or certain event.

Definition 8

Two events cannot occur simultaneously are mutually exclusive events. $A_1, A_2, A_3, \ldots, A_k$ are **mutually exclusive or disjoint events** means that, $A_i \cap A_j = \varnothing$, for $i \neq j$.

Definition 9

Two events are **mutually inclusive** when they can both occur simultaneously.

$A_1, A_2, A_3, \ldots, A_k$ are mutually inclusive means that, $A_i \cap A_j \neq \varnothing$, for $i \neq j$

Illustration

When we roll a die, the sample space $S = \{1,2,3,4,5,6\}$.

(i) Since $\{1, 3\} \cap \{2, 4, 5, 6\} = \varnothing$, the events $\{1,3\}$ and $\{2, 4,5,6\}$ are mutually exclusive events.

(ii) The events $\{1,6,\},\{2,3,5\}$ are mutually exclusive.

(iii) The events $\{2,3,5\},\{5,6\}$ are mutually inclusive, since $\{2, 3, 5\} \cap \{5, 6\} = \{5\} \neq \varnothing$

Definition 10

$A_1, A_2, A_3, \ldots, A_k$ are called **exhaustive events** if, $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k = S$

Definition 11

$A_1, A_2, A_3, \ldots, A_k$ are called **mutually exclusive and exhaustive events** if,

(i) $A_i \cap A_j \neq \varnothing$, for $i \neq j$ (ii) $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k = S$

Illustration 12.4

When a die is rolled, sample space $S = \{1,2,3,4,5,6\}$.
Some of the events are $\{2,3\},\{1,3,5\},\{4,6\},\{6\}$ and $\{1,5\}$.

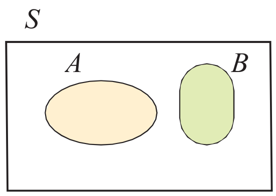
(i) Since $\{2, 3\}\cup\{1, 3, 5\}\cup\{4, 6\} = \{1, 2, 3, 4, 5, 6\} = S$ (sample space), the events $\{2,3\},\{1,3,5\},\{4,6\}$ are exhaustive events.

(ii) Similarly $\{2,3\},\{4,6\}$ and $\{1,5\}$ are also exhaustive events.

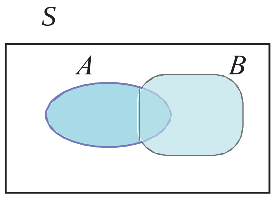
(iii) $\{1,3,5\},\{4,6\},\{6\}$ and $\{1,5\}$ are not exhaustive events.
(Since $\{1, 3, 5\}\cup\{4, 6\}\cup\{6\}\cup\{1, 5\} \neq S$)

(iv) $\{2,3\},\{4,6\}$, and $\{1,5\}$ are mutually exclusive and exhaustive events, since
$\{2,3\}\cap\{4,6\}=\varnothing$, $\{2,3\}\cap\{1,5\}=\varnothing$, $\{4,6\}\cap\{1,5\}=\varnothing$ and $\{2, 3\}\cup \{4, 6\}\cup \{1, 5\} = S$

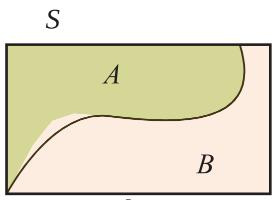
Types of events associated with sample space are easy to visualize in terms of Venn diagrams, **as illustrated below.**



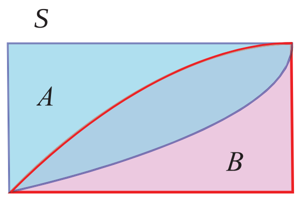
A and *B* are
Mutually exclusive



A and *B* are
Mutually inclusive



A and *B* are
Mutually exclusive
and exhaustive

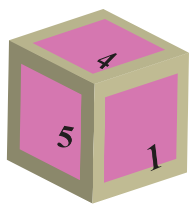


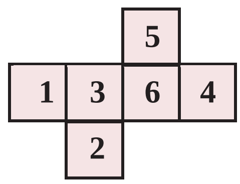
A and *B* are
Mutually inclusive
and exhaustive

Definition 12
The events having the same chance of occurrences are called **equally likely** events.

Example for **equally likely events:** Suppose a fair die is rolled.

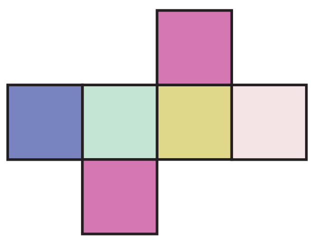
Number on the face	1	2	3	4	5	6
Chance of occurrence	1	1	1	1	1	1

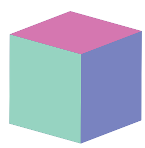




Example for **not equally likely events:** A colour die is shown in figure is rolled.

Colour on the face					
Chance of occurrence	1	1	1	2	1





Similarly, suppose if we toss a coin, the events of getting a head or a tail are equally likely.

Methods to find sample space

Illustration 12.5

Two coins are tossed, the sample space is

(i) $S = \{H,T\}\times\{H,T\} = \{(H,H),(H,T),(T,H),(T,T)\}$ or $\{HH, HT, TH, TT\}$

(ii) If a coin is tossed and a die is rolled simultaneously, then the sample space is

$S = \{H,T\}\times\{1,2,3,4,5,6\} = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$ or

$S = \{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$.

Also one can interchange the order of outcomes of coin and die. The following table gives the sample spaces for some random experiments.

Random Experiment	Total Number of Outcomes	Sample space
Tossing a fair coin	$2^1 = 2$	$\{H, T\}$
Tossing two coins	$2^2 = 4$	$\{HH, HT, TH, TT\}$
Tossing three coins	$2^3 = 8$	$\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$
Rolling fair die	$6^1 = 6$	{1, 2,3, 4, 5,6}
Rolling Two dice or single die two times.	$6^2 = 36$	{(1,1) ,(1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2),(6,3), (6,4), (6,5), (6,6)}
Drawing a card from a pack of 52 playing cards	$52^1 = 52$	Heart ♥ A 2 3 4 5 6 7 8 9 10 J Q K Red in colour Diamond ♦ A 2 3 4 5 6 7 8 9 10 J Q K Red in colour Spade ♠ A 2 3 4 5 6 7 8 9 10 J Q K Black in colour Club ♣ A 2 3 4 5 6 7 8 9 10 J Q K Black in colour

5.3 Mathematical & Statistical Definitions of Probability:

Mathematical or Classical definition of probability

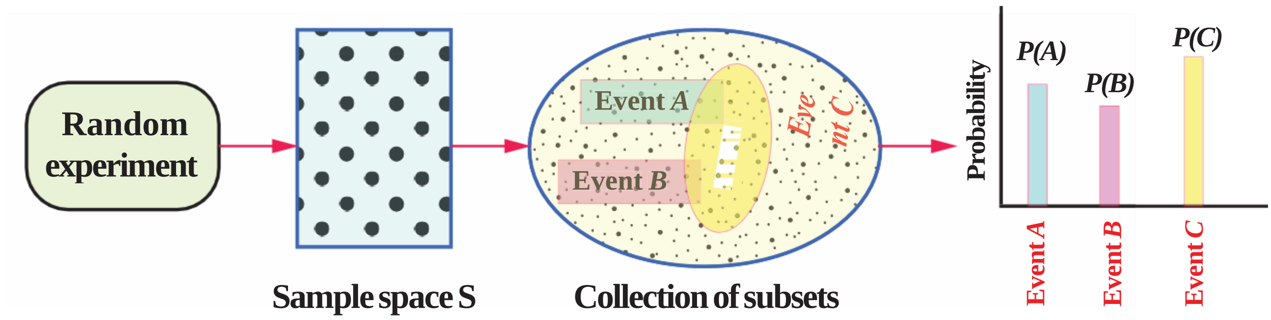
The basic assumption of underlying the classical theory is that the outcomes of a random experiment are equally likely. If there are n exhaustive, mutually exclusive and equally likely outcomes of an experiment and m of them are favorable to an event A, then the mathematical probability of A is defined as the ratio $\frac{m}{n}$. In other words, $P(A) = \frac{m}{n}$.

Definition 13

Let S be the sample space associated with a random experiment and A be an event. Let $n(S)$ and $n(A)$ be the number of elements of S and A respectively. Then the probability of the event A is defined as

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{Number of cases favourable to } A}{\text{Exhaustive number of cases in } S}$$

Every probabilistic model involves an underlying process is shown in the following figure.



The classical definition of probability is limited in its application only to situations where there are a finite number of possible outcomes. It mainly considered discrete events and its methods were mainly combinatorial. This renders it inapplicable to some important random experiments, such as 'tossing a coin until a head appears' which give rise to the possibility of infinite set of outcomes. Another limitation of the classical definition was the condition that each possible outcome is 'equally likely'.

These types of limitations in the classical definition of probability led to the evolution of the modern definition of probability which is based on the concept of sets. It is known an axiomatic approach.

A.N. Kolmogorov

Axiomatic / Statistical approach to Probability

Axioms of probability

Let S be a finite sample space, let $\mathscr{P}(S)$ be the class of events, and let P be a real valued function defined on $\mathscr{P}(S)$ Then $P(A)$ is called probability function of the event A, when the following axioms are hold:

[P_1] For any event A, $P(A) \geq 0$ (Non-negativity axiom)

[P_2] For any two mutually exclusive events $P(A \cup B) = P(A) + P(B)$ (Additivity axiom)

[P_3] For the certain event $P(S) = 1$ (Normalization axiom)

(i) $0 \leq P(A) \leq 1$

(ii) If $A_1, A_2, A_3, \ldots, A_n$ are mutually exclusive events in a sample space S, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots.. \cup A_n) = P(A_1) + P(A_2) + P(A_3) + \ldots.. + P(A_n)$$

Example

If an experiment has exactly the three possible mutually exclusive outcomes A, B, and C, check in each case whether the assignment of probability is permissible.

(i) $P(A)=\frac{4}{7}$, $P(B)=\frac{1}{7}$, $P(C)=\frac{2}{7}$.

(ii) $P(A)=\frac{2}{5}$, $P(B)=\frac{1}{5}$, $P(C)=\frac{3}{5}$.

(iii) $P(A)=0.3$, $P(B)=0.9$, $P(C)=-0.2$.

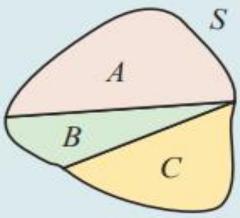
(iv) $P(A)=\frac{1}{\sqrt{3}}$, $P(B)=1-\frac{1}{\sqrt{3}}$, $P(C)=0$.

(v) $P(A)=0.421$, $P(B)=0.527$ $P(C)=0.042$.

Solution

Since the experiment has exactly the three possible mutually exclusive outcomes A, and C, they must be exhaustive events.

$$\Rightarrow S = A\cup B\cup C$$



Therefore, by axioms of probability

$P(A)\geq 0$, $P(B)\geq 0$, $P(C)\geq 0$ and

$P(A\cup B\cup C)=P(A)+P(B)+P(C)=P(S)=1$

(i) Given that $P(A)=\frac{4}{7}\geq 0$, $P(B)=\frac{1}{7}\geq 0$, and $P(C)=\frac{2}{7}\geq 0$

Also $P(S)=P(A)+P(B)+P(C)=\frac{4}{7}+\frac{1}{7}+\frac{2}{7}=1$

Therefore the assignment of probability is permissible.

(ii) Given that $P(A)=\frac{2}{5}\geq 0$, $P(B)=\frac{1}{5}\geq 0$, and $P(C)=\frac{3}{5}\geq 0$

But $P(S)=P(A)+P(B)+P(C)=\frac{2}{5}+\frac{1}{5}+\frac{3}{5}=\frac{6}{5}>1$

Therefore the assignment is not permissible.

(iii) Since $P(C)=-0.2$ is negative, the assignment is not permissible.

(iv) The assignment is permissible because

$P(A)=\frac{1}{\sqrt{3}}\geq 0$, $P(B)=1-\frac{1}{\sqrt{3}}\geq 0$, and $P(C)=0\geq 0$

$P(S)=P(A)+P(B)+P(C)=\frac{1}{\sqrt{3}}+1-\frac{1}{\sqrt{3}}+0=1.$

(v) Even though $P(A)=0.421\geq 0$, $P(B)=0.527\geq 0$, and $P(C)=0.042\geq 0$, the sum of the probability

$P(S)=P(A)+P(B)+P(C)=0.421+0.527+0.042=0.990<1.$

Therefore, the assignment is not permissible.

Example

An integer is chosen at random from the first ten positive integers. Find the probability that it is (i) an even number (ii) multiple of three.

Solution

The sample space is

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}, \quad n(S) = 10$$

Let A be the event of choosing an even number and

B be the event of choosing an integer multiple of three.

$$A = \{2, 4, 6, 8, 10\}, \quad n(A) = 5,$$

$$B = \{3, 6, 9\}, \quad n(B) = 3$$

$$P\text{ (choosing an even integer)} = P(A) = \frac{n(A)}{n(S)} = \frac{5}{10} = \frac{1}{2}.$$

$$P\text{ (choosing an integer multiple of three)} = P(B) = \frac{n(B)}{n(S)} = \frac{3}{10}.$$

Example

Three coins are tossed simultaneously, what is the probability of getting (i) exactly one head (ii) at least one head (iii) at most one head?

Solution:

Notice that three coins are tossed simultaneously = one coin is tossed three times.

The sample space $S = \{H,T\} \times \{H,T\} \times \{H,T\}$

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}, \quad n(S) = 8$$

Let A be the event of getting one head, B be the event of getting at least one head and C be the event of getting at most one head.

$$A = \{HTT, THT, TTH\}; \quad n(A) = 3$$

$$B = \{HTT, THT, TTH, HHT, HTH, THH, HHH\}; \quad n(B) = 7$$

$$C = \{TTT, HTT, THT, TTH\}; \quad n(C) = 4.$$

Therefore the required probabilities are

(i) $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{3}{8}$

(ii) $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{7}{8}$

(iii) $P(C) = \dfrac{n(C)}{n(S)} = \dfrac{4}{8} = \dfrac{1}{2}.$

Example

Suppose a fair die is rolled. Find the probability of getting
(i) an even number (ii) multiple of three.

Solution

Let S be the sample space,
A be the event of getting an even number,
B be the event of getting multiple of three.

Therefore,

$S = \{1, 2, 3, 4, 5, 6\}. \quad \Rightarrow n(S) = 6$

$A = \{2, 4, 6\} \quad \Rightarrow n(A) = 3$

$B = \{3, 6\} \quad \Rightarrow n(B) = 2$

The required probabilities are

(i) P (getting an even number) $= P(A) = \dfrac{n(A)}{n(S)} = \dfrac{3}{6} = \dfrac{1}{2}$

(ii) P (getting multiple of three) $= P(B) = \dfrac{n(B)}{n(S)} = \dfrac{2}{6} = \dfrac{1}{3}$.

Example 12.6

When a pair of fair dice is rolled, what are the probabilities of getting the sum
(i) 7 (ii) 7 or 9 (iii) 7 or 12?

Solution

The sample space $S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$

$$\begin{aligned} S = \{&(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\ &(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\ &(3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\ &(4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\ &(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\ &(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\} \end{aligned}$$

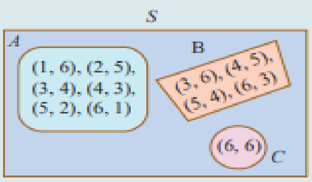
Number of possible outcomes $= 6^2 = 36 = n(S)$

Let A be the event of getting sum 7, B be the event of getting the sum 9 and C be the event of getting sum 12. Then

$A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\} \quad \Rightarrow n(A) = 6$

$B = \{(3,6), (4,5), (5,4), (6,3)\} \quad \Rightarrow n(B) = 4$

$C = \{(6,6)\} \quad \Rightarrow n(C) = 1$



(i) P (getting sum 7) $= P(A)$

$$= \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

(ii) P (getting sum 7 or 9) $= P(A \text{ or } B) = P(A \cup B)$

$$= P(A) + P(B)$$

(Since A and B are mutually exclusive that is, $A \cap B = \varnothing$)

$$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} = \frac{6}{36} + \frac{4}{36} = \frac{5}{18}$$

(iii) P (getting sum 7 or 12) $= P(A \text{ or } C) = P(A \cup C)$

$= P(A) + P(C)$ (since A and C are mutually exclusive)

$$= \frac{n(A)}{n(S)} + \frac{n(C)}{n(S)} = \frac{6}{36} + \frac{1}{36} = \frac{7}{36}.$$

Example

Three candidates X, Y, and Z are going to play in a chess competition to win FIDE (World Chess Federation) cup this year. X is thrice as likely to win as Y and Y is twice as likely as to win Z. Find the respective probability of X, Y and Z to win the cup.

Solution

Let A, B, C be the event of winning FIDE cup respectively by X, Y, and Z this year.

Given that X is thrice as likely to win as Y.

$$A : B \; :: \; 3 : 1. \qquad (1)$$

Y is twice as likely as to win Z

$$B : C :: \; 2 : 1 \qquad (2)$$

From (1) and (2)

$$A : B : C :: \; 6 : 2 : 1$$

$A = 6k$, $B = 2k$, $C = k$, where k is proportional constant.

Probability to win the cup by X is $P(A) = \frac{6k}{9k} = \frac{2}{3}$

Probability to win the cup by Y is $P(B) = \frac{2k}{9k} = \frac{2}{9}$ and

Probability to win the cup by Z is $P(C) = \frac{k}{9k} = \frac{1}{9}$.

5.4 Addition Theorem

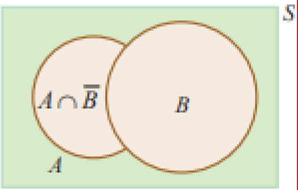
Addition theorem on probability

If A and B are any two events, then

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

Proof

From the diagram,



$$A \cup B = (A \cap \overline{B}) \cup B$$

$$P(A \cup B) = P\left[(A \cap \overline{B}) \cup B\right]$$

$$= P(A \cap \overline{B}) + P(B) \quad \text{(since } (A \cap \overline{B}) \text{ and } B \text{ are mutually exclusive)}$$

$$= [P(A) - P(A \cap B)] + P(B)$$

Therefore, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Note 12.4

The above theorem can be extended to any 3 events.

(i) $P(A \cup B \cup C) = \{P(A) + P(B) + P(C)\}$
$- \{P(A \cap B) + P(B \cap C) + P(C \cap A)\} + P(A \cap B \cap C)\}$

(ii) $P(A \cup B \cup C) = 1 - P(\overline{A \cup B \cup C}) = 1 - P(\overline{A} \cap \overline{B} \cap \overline{C})$

Example

Given that $P(A) = 0.52$, $P(B) = 0.43$, and $P(A \cap B) = 0.24$, find

(i) $P(A \cap \overline{B})$ (ii) $P(A \cup B)$ (iii) $P(\overline{A} \cap \overline{B})$ (iv) $P(\overline{A} \cup \overline{B})$.

Solution

(i) $P(A \cap \overline{B}) = P(A) - P(A \cap B)$

$= 0.52 - 0.24 = 0.28$

$P(A \cap \overline{B}) = 0.28.$

(ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.52 + 0.43 - 0.24$

$P(A \cup B) = 0.71.$

(iii) $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B})$ (By de Morgan's law)

$= 1 - P(A \cup B)$

$= 1 - 0.71 = 0.29.$

(iv) $P(\overline{A} \cup \overline{B}) = P(\overline{A \cap B})$ (By de Morgan's law)

$= 1 - P(A \cap B) = 1 - 0.24$

$= 0.76.$

Example

The probability that a girl, preparing for competitive examination will get a State Government service is 0.12, the probability that she will get a Central Government job is 0.25, and the probability that she will get both is 0.07. Find the probability that (*i*) she will get atleast one of the two jobs (*ii*) she will get only one of the two jobs.

Solution

Let I be the event of getting State Government service and C be the event of getting Central Government job.

Given that $P(I) = 0.12, \quad P(C) = 0.25, \text{ and } \quad P(I \cap C) = 0.07$

(*i*) P (at least one of the two jobs) $= P(I \text{ or } C) = P(I \cup C)$

$$= P(I) + P(C) - P(I \cap C)$$

$$= 0.12 + 0.25 - 0.07 = 0.30$$

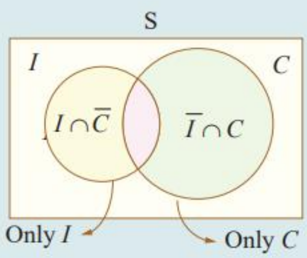
(*ii*) P(only one of the two jobs) $=$ P[only I or only C].

$$= P(I \cap \bar{C}) + P(\bar{I} \cap C)$$

$$= \{P(I) - P(I \cap C)\} + \{P(C) - P(I \cap C)\}$$

$$= \{0.12 - 0.07\} + \{0.25 - 0.07\}$$

$$= 0.23.$$



Conditional Probability

Definition

The **conditional probability** of an event B, assuming that the event A has already happened is denoted by $P(B/A)$ and is defined as

$$\boxed{P(B/A) = \frac{P(A \cap B)}{P(A)}, \quad \text{provided } P(A) \neq 0}$$

Similarly,

$$\boxed{P(A/B) = \frac{P(A \cap B)}{P(B)}, \quad \text{provided } P(B) \neq 0}$$

Example

If $P(A) = 0.6, \quad P(B) = 0.5, \quad \text{and} \quad P(A \cap B) = 0.2$

Find (i) $P(A/B)$ (ii) $P(\bar{A}/B)$ (iii) $P(A/\bar{B})$.

Solution

Given that $P(A) = P(A) = 0.6, \quad P(B) = 0.5, \quad \text{and} \quad P(A \cap B) = 0.2$

(i) $$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.5} = \frac{2}{5}$$

(ii) $$P(\overline{A}/B) = \frac{P(\overline{A} \cap B)}{P(B)}$$

$$= \frac{P(B) - P(A \cap B)}{P(B)}$$

$$= \frac{0.5 - 0.2}{0.5} = \frac{0.3}{0.5} = \frac{3}{5}.$$

(iii) $$P(A/\overline{B}) = \frac{P(A \cap \overline{B})}{P(\overline{B})}$$

$$= \frac{P(A) - P(A \cap B)}{1 - P(B)}$$

$$= \frac{0.6 - 0.2}{1 - 0.5} = \frac{0.4}{0.5} = \frac{4}{5}.$$

Example

A die is rolled. If it shows an odd number, then find the probability of getting 5.

Solution

Sample space $S = \{1, 2, 3, 4, 5, 6\}$.

Let A be the event of die shows an odd number.

Let B be the event of getting 5.

Then, $A=\{1, 3, 5\}$, $B=\{5\}$, and $A \cap B = \{5\}$.

Therefore, $P(A) = \frac{3}{6}$ and $P(A \cap B) = \frac{1}{6}$

$$P(\text{getting } 5 / \text{die shows an odd number}) = P(B/A)$$

$$= \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{6}}{\frac{3}{6}}$$

$$P(B/A) = \frac{1}{3}.$$

5.5 Multiplication Theorem

The probability of the simultaneous happening of two events A and B is given by

$$P(A \cap B) = P(A/B)P(B)$$

or

$$P(A \cap B) = P(B/A)P(A)$$

Independent Events

Events are said to be independent if occurrence or non-occurrence of any one of the event does not affect the probability of occurrence or non-occurrence of the other events.

Definition

Two events A and B are said to be independent if and only if

$$\boxed{P(A \cap B) = P(A) \cdot P(B)}$$

Note

(1) This definition is exactly equivalent to

$$P(A/B) = P(A) \quad \text{if } P(B) > 0$$
$$P(B/A) = P(B) \quad \text{if } P(A) > 0$$

(2) The events $A_1, A_2, A_3, \ldots, A_n$ are mutually independent if

$$P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2) \cdot \cdots \cdot P(A_n).$$

Theorem

If A and B are independent then

(i) $\overline{A}$ and $\overline{B}$ are independent.

(ii) A and $\overline{B}$ are independent.

(iii) $\overline{A}$ and B are also independent.

Example

Two cards are drawn from a pack of 52 cards in succession. Find the probability that both are Jack when the first drawn card is (i) replaced (ii) not replaced

Solution

Let A be the event of drawing a Jack in the first draw,

B be the event of drawing a Jack in the second draw.

Case (i)

Card is replaced

$$n(A) = 4 \quad \text{(Jack)}$$
$$n(B) = 4 \quad \text{(Jack)}$$
$$\text{and} \quad n(S) = 52 \quad \text{(Total)}$$

Clearly the event A will not affect the probability of the occurrence of event B and therefore A and B are independent.

$$P(A \cap B) = P(A) \,.\, P(B)$$
$$P(A) = \frac{4}{52}, \; P(B) = \frac{4}{52}$$
$$P(A \cap B) = P(A) \; P(B)$$
$$= \frac{4}{52} \,.\, \frac{4}{52}$$
$$= \frac{1}{169}.$$

Case (ii)

Card is not replaced

In the first draw, there are 4 Jacks and 52 cards in total. Since the Jack, drawn at the first draw is not replaced, in the second draw there are only 3 Jacks and 51 cards in total. Therefore the first event A affects the probability of the occurrence of the second event B.

Thus A and B are not independent. That is, they are dependent events.

$$\text{Therefore, } P(A \cap B) = P(A) \,.\, P(B/A)$$
$$P(A) = \frac{4}{52}$$

$$P(B/A) = \frac{3}{51}$$

$$P(A \cap B) = P(A) \,.\, P(B/A)$$

$$= \frac{4}{52} \cdot \frac{3}{51}$$

$$= \frac{1}{221} \,.$$

Example

A coin is tossed twice. Events E and F are defined as follows

E= Head on first toss, F = Head on second toss. Find

(i) $P(E \cup F)$ (ii) $P(E / F)$

(iii) $P(\overline{E} / F)$. (iv) Are the events E and F independent?

Solution

The sample space is $S = \{H, T\} \times \{H, T\}$

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

and $E = \{(H, H), (H, T)\}$

$$F = \{(H, H), (T, H)\}$$

$$E \cup F = \{(H, H), (H, T), (T, H)\}$$

$$E \cap F = \{(H, H)\}$$

(i) $$P(E \cup F) = P(E) + P(F) - P(E \cap F) \quad \text{or} \quad \left(= \frac{n(E \cup F)}{n(S)} \right)$$

$$= \frac{2}{4} + \frac{2}{4} - \frac{1}{4} = \frac{3}{4} \,.$$

(ii) $$P(E / F) = \frac{P(E \cap F)}{P(F)} = \frac{(1/4)}{(2/4)} = \frac{1}{2}$$

(iii) $$P(\overline{E} / F) = \frac{P(\overline{E} \cap F)}{P(F)}$$

$$= \frac{P(F) - P(E \cap F)}{P(F)}$$

$$= \frac{(2/4) - (1/4)}{(2/4)}$$

$$= \frac{1}{2}$$

(iv) Are the events E and F independent?

We have $P(E \cap F) = \frac{1}{4}$

$$P(E) = \frac{2}{4}, \quad P(F) = \frac{2}{4}$$

$$P(E)\,P(F) = \frac{2}{4}.\frac{2}{4} = \frac{1}{4}$$

$$\Rightarrow P(E \cap F) = P(E)\,.\,P(F)$$

Therefore E and F are independent events.

Note

Independent events is a property of probability but mutual exclusiveness is a set-theoretic property. Therefore independent events can be identified by their probabilities and mutually exclusive events can be identified by their events.

Theorem

Suppose A and B are two events, such that $P(A) \neq 0$, $P(B) \neq 0$.

(1) If A and B are mutually exclusive, they cannot be independent.

(2) If A and B are independent they cannot be mutually exclusive. (Without proof)

Example

If A and B are two independent events such that $P(A) = 0.4$ and $P(A \cup B) = 0.9$. Find $P(B)$.

Solution

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = P(A) + P(B) - P(A).P(B) \qquad \text{(since } A \text{ and } B \text{ are independent)}$$

That is, $0.9 = 0.4 + P(B) - (0.4)\,P(B)$

$$0.9 - 0.4 = (1 - 0.4)\,P(B)$$

Therefore, $P(B) = \frac{5}{6}$.

Example

An anti-aircraft gun can take a maximum of four shots at an enemy plane moving away from it. The probability of hitting the plane in the first, second, third, and fourth shot are respectively 0.2, 0.4, 0.2 and 0.1. Find the probability that the gun hits the plane.

Solution

Let H_1, H_2, H_3 and H_4 be the events of hitting the plane by the anti-aircraft gun in the first second, third and fourth shot respectively.

Let H be the event that anti-aircraft gun hits the plane. Therefore $\overline{H}$ is the event that the plane is not shot down. Given that

$$P(H_1) = 0.2 \quad \Rightarrow P(\overline{H}_1) = 1 - P(H_1) = 0.8$$

$$P(H_2)=0.4 \quad \Rightarrow P(\bar{H}_2)=1-P(H_2)=0.6$$

$$P(H_3)=0.2 \quad \Rightarrow P(\bar{H}_3)=1-P(H_3)=0.8$$

$$P(H_4)=0.1 \quad \Rightarrow P(\bar{H}_4)=1-P(H_4)=0.9$$

The probability that the gun hits the plane is

$$P(H)=1-P(\bar{H})=1-P(\overline{H_1\cup H_2\cup H_3\cup H_4})$$

$$=1-P\left(\bar{H}_1\cap\bar{H}_2\cap\bar{H}_3\cap\bar{H}_4\right)$$

$$=1-P(\bar{H}_1)P(\bar{H}_2)P(\bar{H}_3)P(\bar{H}_4)$$

$$=1-(0.8)(0.6)(0.8)(0.9)=1-0.3456$$

$$P(H)=0.6544$$

Example

Urn-I contains 8 red and 4 blue balls and urn-II contains 5 red and 10 blue balls. One urn is chosen at random and two balls are drawn from it. Find the probability that both balls are red.

Solution

Let A_1 be the event of selecting urn-I and A_2 be the event of selecting urn-II.

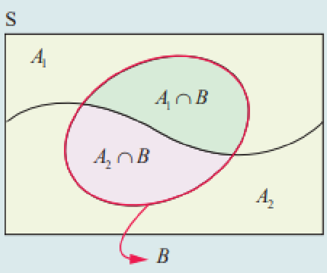
Let B be the event of selecting 2 red balls.

We have to find the total probability of event B. That is, $P(B)$.

Clearly A_1 and $A_2 A_1$ are mutually exclusive and exhaustive events.

	Red balls	Blue balls	Total
Urn-I	8	4	12
Urn-II	5	10	15
Total	13	14	27

We have



$$P(A_1)=\frac{1}{2}, \quad P(B/A_1)=\frac{8c_2}{12c_2}=\frac{14}{33}$$

$$P(A_2)=\frac{1}{2}, \quad P(B/A_2)=\frac{5c_2}{15c_2}=\frac{2}{21}$$

We know

$$P(B)=P(A_1).P(B/A_1)+P(A_2).P(B/A_2)$$

$$P(B)=\frac{1}{2}.\frac{14}{33}+\frac{1}{2}.\frac{2}{21}=\frac{20}{77}.$$

@@@ End of **UNIT-V** @@@