(b) $AR = \frac{R}{q} = \frac{-\frac{q^2}{2} + 50q}{q} = -\frac{q}{2} + 50$

(c) $MR = \frac{dR}{dq} = \frac{d}{dq}\left(\frac{-q^2}{2} + 50q\right) = \frac{-2q}{2} + 50 = -q + 50.$

EXERCISE 6.7

1. Given the demand curve $P = 16 - D^2$, find the total revenue curve and marginal revenue curve when $D = 1$.
2. If $P = 12 + 8Q - Q^2$ is the demand function, find the total and marginal revenue functions.
3. Given the demand function $P = 40 - 2q$, find the total and marginal revenue functions.
4. Let the demand function for a commodity be $P = 10 - 2D$, where P is the price and D is the quantity demanded. Find AR and MR.

Find AR and MR for the following functions :

5. $R = 2x - 3x^2$
6. $R = 3 + 5x - x^2$
7. $R = 8x - 2x^2$
8. $R = 100x - x^2 - x^3$
9. $X = 20 - 2P^2$
10. $X = 2P - P^2$.

CHAPTER 7

INTEGRAL CALCULUS

MEANING

In calculus, the reverse or inverse process of "Differentiation" is called "*Indefinite Integration*" or "*Integral Calculus*" or "*Anti differentiation*". The process of Integration is finding the function whose Derivative or Differential is given.

Example:

We have seen that, differentiation means finding the rate at which a variable quantity is changing (i.e., Derivative). If $y = f(x)$, we can find out the rate of change of y with respect to x by differentiating the function and thereby obtaining $\frac{dy}{dx}$ or $f'(x)$.

Therefore, if total Utility Function is $U = q^2$, where, U is the total utility and q is the quantity consumed, then $\frac{dU}{dq} = 2q$ is the rate of change of total utility (U) with respect to quantity consumed (q) and is the marginal utility function.

Now, if the marginal utility function is 2q, what is the total utility function? The total utility function $U=q^2$. The problem is reversed here. Because, we are now given the marginal utility function and we have to find the total utility function. In other words, we are given the rate of change of quantities and we have to calculate the total effects of the change (i.e., function). This reverse process involves the summation of the differences. Process of this calculation is called "Integration".

Differentiation

If Total Utility Function is q^2, then the Marginal Utility Function is 2q.

Integration (Indefinite Integration)

If the Marginal Utility Function is 2q, then the Total Utility Function is q^2.

That is, if derivative is given, our problem is to find out the function.

Definition

If the Differential co-efficient of F(x) with respect to x is f(x), then an Integral of f(x) with respect to x is F(x).

Notation

If $\frac{d}{dx}$ F(x) = f(x), then the usual notation for the indefinite integral is,

$$\int f(x)\,dx \text{ or } \int y\,dx = F(x).$$

where,

(i) the symbol $\int$ is called Integral sign.
(ii) f(x) is called the "Integrand" or the function to be integrated.
(iii) dx suggests that the operation of integration is to be with respect to the variable x.

The statement, "Evaluate $\int f(x)\,dx$" means *"Find the antiderivative or Integral of f(x)"*.

SOME BASIC RULES OF INTEGRATION

In fact, every differentiation rule has a counterpart in *Integration.*

Rule I

$\int dx = x + c$

Rule II

If a function is multiplied by a constant, the integral of that function is also multipled by that same constant.

$\int K\,dx = K \int dx = Kx + C$, where K is constant.

Examples:

1. $\int 5\,dx = 5 \int dx = 5x + C$
2. $\int 7\,dx = 7 \int dx = 7x + C$

Rule III

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$$

(i.e., the integral of x^n with reference to x)

Examples:

1. $\int x^5\,dx = \frac{x^{5+1}}{5+1} + C = \frac{x^6}{6} + C$
2. $\int x\,dx = \frac{x^{1+1}}{1+1} + C = \frac{x^2}{2} + C$
3. $\int 1\,dx = \int x^0\,dx = x + C$
4. $\int \frac{1}{x^7}\,dx = \int x^{-7}\,dx = \frac{x^{-7+1}}{-7+1} + C = \frac{x^{-6}}{-6} + C$
5. $\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{x^{1/2+1}}{\frac{1}{2}+1} + C = \frac{x^{3/2}}{\frac{3}{2}} + C$
6. $\int x^{-9}\,dx = \frac{x^{-9+1}}{-9+1} + C = \frac{x^{-8}}{-8} + C = -\frac{1}{8} x^{-8} + C$

Rule IV

Integral of Sum or Difference

Integral of sum or difference of a number of functions is equal to the sum or difference of the Integrals of the separate functions.

In symbols

$$\int (dx_1 + dx_2 + \ldots + dx_n) = \int dx_1 + \int dx_2 + \ldots\ldots + \int dx_n + C.$$

Or

$$\int (dx_1 - dx_2 - \ldots - dx_n) = \int dx_1 - \int dx_2 - \ldots\ldots - \int dx_n + C.$$

Examples:

1. $\int (x^3 - x + 1)\,dx = \int x^3\,dx - \int x\,dx + \int 1\,dx = \frac{x^4}{4} - \frac{x^2}{2} + x + C.$

2. $\int \left(x^5 + \frac{1}{x^7} - x - 1\right) dx$

$= \int x^5\,dx + \int \frac{1}{x^7}\,dx - \int x\,dx - \int 1\,dx$

$= \left(\frac{x^6}{6}\right) + \left(\frac{x^{-7+1}}{-7+1}\right) - \left(\frac{x^2}{2}\right) - x + C$

$= \frac{x^6}{6} + \frac{x^{-6}}{-6} - \frac{x^2}{2} - x + C.$

3. $\int (8x^3 - 3x^2 + x - 1)\,dx$

$= \int 8x^3\,dx - \int 3x^2\,dx + \int x\,dx - \int 1\,dx$

$= 8\int x^3\,dx - 3\int x^2\,dx + \int x\,dx - 1\int dx$

$= 8\frac{x^4}{4} - 3\frac{x^3}{3} + \frac{x^2}{2} - x + C = 2x^4 - x^3 + \frac{x^2}{2} - x + C.$

Rule V

Integral of a Multiple by a Constant

If a function is multiplied by a constant number, this number will remain a multiple of the integral of the function.

Examples:

1. $\int 4x^8\,dx = 4\int x^8\,dx = 4\frac{x^{8+1}}{8+1} + C = 4\frac{x^9}{9} + C$
2. $\int 4x^3\,dx = 4\int x^3\,dx = 4\frac{x^{3+1}}{3+1} + C = 4\frac{x^4}{4} + C = x^4 + C$

3. $\int 4x\,dx = 4\int x\,dx = 4\frac{x^{1+1}}{1+1} + C = 4\frac{x^2}{2} + C = 2x^2 + C$

4. $\int 3x^2dx = 3\int x^2dx = 3\frac{x^{2+1}}{2+1} + C = 3\frac{x^3}{3} + C = x^3 + C$

5. $\int 5x^4dx = 5\int x^4\,dx = 5\frac{x^5}{5} + C = x^5 + C$

6. $\int 6x^5\,dx = 6\int x^5\,dx = 6\frac{x^6}{6} + C = x^6 + C$

7. $\int 7x^6dx = 7\int x^6\,dx = 7\frac{x^7}{7} + C = x^7 + C$.

Rule VI

Integration by Substitution

In the case of a product or quotient of two differentiable functions of x, it may be possible to express them as a constant multiple of another function f(u) and its derivative is $\frac{du}{dx}$,

i.e., $\int f(x)\,dx = \int\left[f(u)\frac{du}{dx}\right]dx = \int f(u)\,.\,du = F(u) + C$

The substitution method is the counterpart of the "Chain Rule" in Differential Calculus.

Examples :

(A) Product

1. Evaluate $\int 4x^2(x^3+5)^3\,dx$.

Solution: Let $U = x^3 + 5$

Hence, $\frac{du}{dx} = 3x^2$

$3x^2\,dx = du$

Therefore, $dx = \frac{du}{3x^2}$

Hence, on substitution we have,

$\int 4x^2(x^3+5)^3\,dx = \int 4x^2\,.\,u^3\,dx$ ..... (Since $u = x^3 + 5$)

$= \int 4x^2\,.\,u^3\,.\,\frac{du}{3x^2}$ (Since $dx = \frac{du}{3x^2}$)

$= \int\frac{4}{3}u^3\,du = \frac{4}{3}\int u^3\,du$

$= \frac{4}{3}\,\frac{u^{3+1}}{3+1} + C = \frac{4}{3}\,\frac{u^4}{4} + C$

$= \frac{1}{3}u^4 + C$

$= \frac{1}{3}(x^3+5)^4 + C$ .... (Since $u = x^3 + 5$)

$= \frac{(x^3+5)^4}{3} + C.$

2. Evaluate $\int 9x^4(x^5+7)^8\,dx$.

Solution : Let $u = x^5 + 7$

Therefore, $\frac{du}{dx} = 5x^4$

$5x^4\,dx = du$

Therefore, $dx = \frac{du}{5x^4}$

Hence, on substitution we have,

$\int 9x^4(x^5+7)^8\,dx = \int 9x^4.\,u^8\,dx$ ..... (Since $u = x^5 + 7$)

$= \int 9x^4.\,u^8\,\frac{du}{5x^4}$ ...... (Since $dx = \frac{du}{5x^4}$)

$= \int\frac{9}{5}u^8\,du$

$= \frac{9}{5}\int u^8\,du = \frac{9}{5}\,\frac{u^{8+1}}{8+1} + C$

$= \frac{9}{5}\,\frac{u^9}{9} + C = \frac{u^9}{5} + C$

$= \frac{(x^5+7)^9}{5} + C$ ..... (Since $u = x^5 + 7$).

3. Evaluate $\int 21x^6(x^7+1)^2\,dx$.

Solution: Let $u = x^7 + 1$

Therefore, $\frac{du}{dx} = 7x^6$

$7x^6\,dx = du$

$dx = \frac{du}{7x^6}$

Hence, on substitution we have,

$\int 21x^6(x^7+1)^2dx = \int 21x^6\,.\,u^2\,dx$ .... (Since $u = x^7 + 1$)

$= \int 21x^6\,.\,u^2\,.\,\frac{du}{7x^6}$ ... (Since $dx = \frac{du}{7x^6}$)

$= 3\int u^2\,du$

$$= 3\frac{u^{2+1}}{2+1} + C = 3\ \frac{u^3}{3} + C$$

$$= u^3 + C = (x^7 + 1)^3 + C \ldots\ldots \text{(Since } u = x^7 + 1\text{)}.$$

(B) Quotient

1. Evaluate $\int \frac{3x}{(x^2-2)^2}dx.$

Solution : Let $u = x^2 - 2$

Therefore, $\frac{du}{dx} = 2x$

$2x\,dx = du$

$dx = \frac{du}{2x}$

Hence, on substitution we have,

$$\int \frac{3x}{(x^2-2)^2}dx = \int \frac{3x}{u^2}dx \ldots\ldots\ldots \text{(Since } u = x^2 - 2\text{)}$$

$$= \int 3x.\, u^{-2}\,dx$$

$$= \int 3x\,.\,u^{-2}\,\frac{du}{2x} \ldots\ldots\ldots \left(\text{Since } dx = \frac{du}{2x}\right)$$

$$= \int \frac{3}{2}u^{-2}\,du$$

$$= \frac{3}{2}\int u^{-2}\,du = \frac{3}{2}\cdot\frac{u^{-2+1}}{-2+1} + C.$$

$$= \frac{3}{2}\cdot\frac{u^{-1}}{(-1)} + C$$

$$= -\frac{3}{2}(x^2-2)^{-1} + C. \ \ldots\ldots \text{(Since } u = x^2 - 2\text{)}. \ \ldots$$

2. Evaluate $\int \frac{8x}{(x^2-5)^3}dx.$

Solution :

Let $u = x^2 - 5$

$\frac{du}{dx} = 2x$

$2x\,dx = du$

$dx = \frac{du}{2x}$

Hence, on substitution we have,

$$\int \frac{8x}{(x^2-5)^3}dx = \int \frac{8x}{u^3}dx$$

$$= \int \frac{8x}{u^3}\times\frac{du}{2x} \ldots\ldots\ldots \left(\text{Since } dx = \frac{du}{2x}\right)$$

$$= 4\int \frac{du}{u^3} = 4\int u^{-3}\,du = 4\,.\,\frac{u^{-3+1}}{-3+1} + C.$$

$$= 4\,.\,\frac{u^{-2}}{-2} + C = -2u^{-2} + C = -2\frac{1}{u^2} + C$$

$$= \frac{-2}{(x^2-5)^2} + C$$

3. Evaluate $= \int \frac{40x^3}{(20x^4+2)^4}dx.$

Solution:

Let $u = 20x^4 + 2$

$\frac{du}{dx} = 80x^3$

$80x^3\,dx = du$

$dx = \frac{du}{80x^3}$

Hence, on substitution we have,

$$\int \frac{40x^3}{(20x^4+2)^4}dx = \int \frac{40x^3}{u^4}\,.\,dx$$

$$= \int \frac{40x^3}{u^4}\times\frac{du}{80x^3} = \frac{1}{2}\int u^{-4}\,du$$

$$= \frac{1}{2}\ \frac{u^{-3}}{-3} + C = \frac{1}{-6}\ u^{-3} + C$$

$$= -\frac{1}{6}\cdot\frac{1}{u^3} + C = -\frac{1}{6u^3} + C$$

$$= -\frac{1}{6\,(20x^4+2)^3} + C \ldots \text{(Since } u = 20x^4 + 2\text{)}.$$

Rule VII

Logarithmic Function Rule

In differentiation, we have seen that, if $y = \log x$, then,

$$\frac{dy}{dx} = \frac{1}{x}$$

Thus, in Integration, we shall define that,

$$\int \frac{1}{x}\,dx = \log x + C$$

where, C is a constant.

Examples :

1. $\int \frac{2}{x} dx = 2 \int \frac{1}{x} dx = 2 \log x + C.$

2. $\int \frac{1}{x+1} dx$

Substitute $z = x + 1$

$$\frac{dz}{dx} = 1$$

$$dz = dx$$

$$\int \frac{1}{z} dz = \log z + C$$

$$= \log (x + 1) + C.$$

3. $\int \frac{x}{x^2 + 1} dx$

Substitute $x^2 + 1 = z$

$$z = x^2 + 1$$

$$\frac{dz}{dx} = 2x$$

$$2x\, dx = dz$$

$$x\, dx = \frac{dz}{2}$$

$$\int \frac{x}{x^2+1} dx = \int \frac{dz}{2z} = \frac{1}{2} \int \frac{dz}{z} \; \ldots \left(\text{sine } x dx = \frac{dz}{2} \text{ and } x^2 + 1 = z \right)$$

$$= \frac{1}{2} \int \frac{1}{z} . dz$$

$$= \frac{1}{2} \log z + C \ldots \left(\int \frac{1}{x} = \log x + C \right)$$

$$= \frac{1}{2} \log (x^2 + 1) + C.$$

Rule VIII

Exponential Function Rule

In differentiation, we have seen that if $y = e^x$, then,

$$\frac{dy}{dx} = e^x$$

Thus, in Integration, we shall define that,

$$y = \int e^x dx = e^x + C.$$

Examples :

1. $\int e^{x+3} dx = e^{x+3} + C.$

2. $\int \left(e^x + \frac{1}{x^3} \right) dx = \int e^x dx + \int \frac{1}{x^3} dx = \int e^x dx + \int x^{-3} dx$

$$= e^x + \frac{x^{-3+1}}{-3+1} + C$$

$$= e^x + \frac{x^{-2}}{-2} + C.$$

Rule IX

$= \int e^{ax} dx$ .... [Substitute $ax = y$

$= \int e^y \frac{dy}{a}$ or $y = ax$, $\therefore \frac{dy}{dx} = a$

$= \frac{1}{a} \int e^y dy$ Therefore, $a\, dx = dy$

$= \frac{1}{a} e^y + C$ or $dx = \frac{dy}{a}$]

$= \frac{1}{a} e^{ax} + C$

Therefore, $\boxed{\int e^{ax} dx = \frac{1}{a} e^{ax} + C}$

Examples :

1. $\int e^{3x} dx = \frac{1}{3} e^{3x} + C$

2. $\int e^{7x} dx = \frac{1}{7} e^{7x} + C$

Rule X

$\int a^{kx} dx$

$= \int e^{k(\log a) x} dx$ ..... [Any Number $= e^{\log (\text{that Number})}$

Therefore,

$a^{kx} = e^{\log (a^{kx})} = e^{kx \log a} = e^{k(\log a) x}$]

$= \frac{1}{k. \log a} . e^{k(\log a) x} + C$ ......

$\left[\text{since } \int e^{ax} dx = \frac{1}{a} . e^{ax} + C \right]$

$$= \frac{a^{kx}}{k \log a} + C \quad [\text{since } e^{k(\log a)x} = a^{kx}]$$

Therefore, $$\boxed{\int a^{kx}\, dx = \frac{a^{kx}}{k \log a} + C}$$

Examples :

1. $\int 5^{7x}\, dx = \frac{5^{7x}}{7 \log 5} + C$

2. $\int 127^{3x}\, dx = \frac{127^{3x}}{3 \log 127} + C$

Rule XI

$\int (ax + b)^n\, dx$

Let $y = ax + b$

$\frac{dy}{dx} = a$

$dy = adx$

$adx = dy$

$$\boxed{dx = \frac{dy}{a}}$$

Therefore, $\int (ax + b)^n\, dx = \int y^n . \frac{dy}{a} = \frac{1}{a} \int y^n\, dy$

$$= \frac{1}{a} \frac{y^{n+1}}{n+1} + C = \frac{1}{a} \frac{(ax + b)^{n+1}}{n+1} + C$$

Therefore, $$\boxed{\int (ax + b)^n\, dx = \frac{1}{a} \frac{(ax + b)^{n+1}}{n+1} + C}$$

Examples :

1. $\int (5x + 2)^7\, dx = \frac{1}{5} \frac{(5x + 2)^8}{8} + C.$

2. $\int (32x + 16)^{15}\, dx = \frac{1}{32} \frac{(32x + 16)^{16}}{16} + C.$

Rule XII

$\int x^{n-1} (ax^n + b)^m\, dx$

Let $y = ax^n + b$

$\frac{dy}{dx} = nax^{n-1}$

Therefore, $dy = n\, ax^{n-1}\, dx$

$n\, ax^{n-1}\, dx = dy$

$$\boxed{x^{n-1}\, dx = \frac{dy}{na}}$$

Therefore, $\int x^{n-1} (ax^n + b)^m\, dx = \int \frac{y^m}{na} dy = \frac{1}{na} \int y^m\, dy$

$$= \frac{1}{na} \frac{y^{m+1}}{m+1} + C = \frac{1}{na} \frac{(ax^n + b)^{m+1}}{m+1} + C.$$

Therefore, $$\boxed{\int x^{n-1} (ax^n + b)^m\, dx = \frac{1}{na} \frac{(ax^n + b)^{m+1}}{m+1} + C.}$$

Examples :

1. $\int x^5 (8x^6 + 7)^{15}\, dx = \frac{1}{6 \times 8} \frac{(8x^6 + 7)^{16}}{16} + C$

$$= \frac{1}{48} \frac{(8x^6 + 7)^{16}}{16} + C.$$

2. $\int x^8 (7x^9 + 15)^{32}\, dx = \frac{1}{63} \frac{(7x^9 + 15)^{33}}{33} + C.$

Aliter :

Let $7x^9 + 15 = y$

$y = 7x^9 + 15$

$\frac{dy}{dx} = 63x^8$

$63x^8\, dx = dy$

$x^8\, dx = \frac{dy}{63}$

Therefore, $\int x^8 (7x^9 + 15)^{32}\, dx = \int y^{32} \frac{dy}{63} = \frac{1}{63} \int y^{32}\, dy$

$$= \frac{1}{63} \frac{y^{33}}{33} + C = \frac{1}{63} \frac{(7x^9 + 15)^{33}}{33} + C.$$

The following results will be useful for finding the Integrals of Trigonometric Functions :

Table 7.1 : Integrals of Trigonometric Functions

Sl. No.	Derivative	Integration
1	$Y = \frac{d}{dx} \sin x = \cos x$	$\int \cos x \, dx = \sin x$
2	$Y = \frac{d}{dx} \cos x = -\sin x$	$\int \sin x \, dx = -\cos x$
3	$Y = \frac{d}{dx} \tan x = \sec^2 x$	$\int \sec^2 x \, dx = \tan x$
4	$Y = \frac{d}{dx} \cot x = -\mathrm{cosec}^2 x$	$\int \mathrm{cosec}^2 x \, dx = -\cot x.$
5	$Y = \frac{d}{dx} \sec x = \sec x \tan x$	$\int \sec x \tan x \, dx = \sec x$
6	$Y = \frac{d}{dx} \mathrm{cosec}\, x = -\mathrm{cosec}\, x \cot x$	$\int \mathrm{cosec}\, x \cot x \, dx = -\mathrm{cosec}\, x.$

EXERCISE 7.1

Evaluate the following :

1. $\int (3x^2 - 5x + 1)\, dx$
2. $\int \left(\frac{3}{x^2} + \frac{1}{x} - 5\right) dx$
3. $\int (4x^3 + 4)\, dx$
4. $\int \frac{1}{x \log x}\, dx$
5. $\int \left(x + \frac{1}{x}\right)^2 dx$
6. $\int 4x\sqrt{x^2 + 8}\; dx$
7. $\int (4 - 3x)^5\, dx$
8. $\int \left(5e^x + \frac{3}{x^2}\right) dx$
9. $\int \frac{3x^2 + 4x - 5}{\sqrt{x}}\, dx$
10. $\int x^2 (1 - x)^2\, dx$

DEFINITE INTEGRATION OR DEFINITE INTEGRAL

We found that finding the antiderivative of a function achieved an indefinite result (i.e., no definite numerical values) and so we called this process as "Indefinite Integration". Now, we have to use the other property of the integral to find the area between two curves and the area between the curve and the X – axis, we shall achieve a definite numerical result i.e., a

number or a value independent of the constant 'C' and not a function as for the Indefinite Integral. Therefore, we call this process as "Definite Integration".

Notation for Definite Integration

$$\int_a^b y\, dx \quad \text{or} \quad \int_a^b f(x)\, dx$$

It is to be read as "the Definite Integral" of y or f (x) from x = a to x = b. Then the value of Definite Integral from a to b is written as,

$$\int_a^b y dx \quad \text{or} \quad \int_a^b f(x)\, dx = [F(x)]_a^b = F(b) - F(a) \ldots b > a$$

where,

$\int_a^b y dx$ or $\int_a^b f(x)\, dx$ is the Definite Integral of y or f(x) and is the area bound by y = f (x), the X–axis and the curves x = a and x=b, a is called the lower limit of the Integral and b is called the upper limit of the Integral.

Examples:

1. $\int_1^2 x^2\, dx = \left[\frac{x^{2+1}}{2+1}\right]_1^2 = \left[\frac{x^3}{3}\right]_1^2 = \left[\frac{2^3}{3} - \frac{1^3}{3}\right] = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$

2. $\int_1^2 x^3\, dx = \left[\frac{x^{3+1}}{3+1}\right]_1^2 = \left[\frac{x^4}{4}\right]_1^2 = \left[\frac{2^4}{4} - \frac{1^4}{4}\right] = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}.$

3. $\int_2^3 3x dx = \left[3\, \frac{x^{1+1}}{1+1}\right]_2^3 = \left[3\, \frac{x^2}{2}\right]_2^3$

$= \left[3\, \frac{3^2}{2} - 3\, \frac{2^2}{2}\right] = \frac{27}{2} - \frac{12}{2} = \frac{15}{2}.$

4. $\int_2^3 4x dx = \left[4\, \frac{x^{1+1}}{1+1}\right]_2^3 = \left[4\, \frac{x^2}{2}\right]_2^3$

$= \left[4\, \frac{3^2}{2} - 4\, \frac{2^2}{2}\right] = \frac{36}{2} - \frac{16}{2} = \frac{20}{2} = 10.$

5. $\int_2^3 (x^2+5x+7)dx = \left[\frac{x^{2+1}}{2+1} + \frac{5x^{1+1}}{1+1} + 7x\right]_2^3$

$= \left[\frac{x^3}{3} + \frac{5x^2}{2} + 7x\right]_2^3$

$= \left[\frac{3^3}{3} + 5\,\frac{3^2}{2} + 7(3)\right] - \left[\frac{2^3}{3} + 5\,\frac{2^2}{2} + 7\,(2)\right]$

$= \left[\frac{27}{3} + \frac{45}{2} + 21\right] - \left[\frac{8}{3} + \frac{20}{2} + 14\right]$

$= \left[\frac{54+135+126}{6}\right] - \left[\frac{16+60+84}{6}\right]$

$= \frac{315}{6} - \frac{160}{6} = \frac{155}{6} = 25\frac{5}{6}.$

6. $\int_1^2 (x^3-2x-3)\,dx = \left[\frac{x^{3+1}}{3+1} - 2\,\frac{x^{1+1}}{1+1} - 3x\right]_1^2 = \left[\frac{x^4}{4} - 2\,\frac{x^2}{2} - 3x\right]$

$= \left[\frac{2^4}{4} - 2\,\frac{2^2}{2} - 3\,(2)\right] - \left[\frac{1^4}{4} - 2\,\frac{(1)^2}{2} - 3(1)\right]$

$= \left[\frac{16}{4} - \frac{8}{2} - 6\right] - \left[\frac{1}{4} - \frac{2}{2} - 3\right]$

$= (4-4-6) - \left(\frac{1}{4} - 1 - 3\right)$

$= 4-4-6-\frac{1}{4}+1+3 = 8 - 10\frac{1}{4} = -2\frac{1}{4}.$

EXERCISE 7.2

1. Evaluate $\int_1^2 (3x^2-2x+1)\,dx$. 2. Evaluate $\int_2^3 (x+4)^3\,dx$.

3. Prove that $\int_0^2 (2x+7)\,dx = 18$. 4. Show that $\int_1^2 (x^2+4x+3)\,dx =$ [illegible]

5. Show that $\int_0^3 (x^2-11x-2)\,dx = -\frac{93}{2}$.

6. Prove that $\int_0^1 (2x^3+5)\,dx = 5.5$.

AREA BETWEEN TWO CURVES

Let $Y_1 = f(x)$ be the curve BOS and $Y_2 = g(x)$ be the curve BES.

Suppose they intersect at B and S where 'X' co–ordinates are OD = a and ON = b.

The above statement is illustrated in the Figure 7.1.

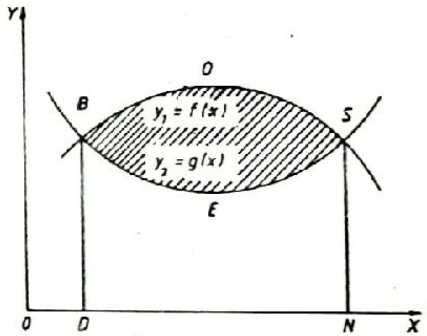

Fig. 7.1 : Area Between Two Curves

We have to find the area BOSE, which is equal to the area DBOSN minus DBESN.

Therefore, area BOSE $= \int_a^b f(x)\,dx - \int_a^b g(x)dx$

Area BOSE $= \int_a^b [f(x) - g(x)]\,dx.$

Examples :

1. Find the area beneath the curve $y = x^5$ between $x = 2$ and $x = 3$.

Solution :

Required area $= \int_2^3 x^5\,dx = \left[\frac{x^{5+1}}{5+1}\right]_2^3 = \left[\frac{x^6}{6}\right]_2^3$

$= \frac{(3)^6}{6} - \frac{(2)^6}{6} = \frac{729}{6} - \frac{64}{6} = \frac{729-64}{6} = \frac{665}{6} = 110.83.$

2. Find the area included between the two parabolas $y^2 = 4x$ and $x^2 = 4y$.

Solution:

$$y^2 = 4x \quad \text{---(1)}$$
$$y = \sqrt{4x}$$
$$x^2 = 4y \quad \text{---(2)}$$
$$4y = x^2$$
$$y = \frac{x^2}{4}$$

Substituting the value of y (i.e., $y = \frac{x^2}{4}$) in equation (1)

$$\left(\frac{x^2}{4}\right)^2 = 4x$$
$$\frac{x^4}{16} = 4x$$
$$x^4 = 16 \times 4x \text{ (cross multiplication)}$$
$$x^4 = 64x$$
$$x^4 - 64x = 0$$
$$x(x^3 - 64) = 0$$

i) $x = 0$ ii) $x^3 - 64 = 0$

$$x^3 = 64$$
$$x = \sqrt[3]{64} = 4$$

Limits are $x = 0$ and $x = 4$

$$\text{The required area} = \int_0^4 \left(\sqrt{4x} - \frac{x^2}{4}\right) dx$$
$$= \int_0^4 \left[\left(\sqrt{4}\cdot\sqrt{x}\right) - \frac{x^2}{4}\right] dx$$
$$= \int_0^4 \left[\left((2.x^{1/2}) - \frac{x^2}{4}\right)\right] dx$$
$$= \left[\left(2 \cdot \frac{x^{3/2}}{\frac{3}{2}}\right) - \frac{x^3}{4 \times 3}\right]_0^4 = \left[\left(2.x^{3/2} \cdot \frac{2}{3}\right) - \frac{x^3}{12}\right]_0^4$$

$$= \left[\left(2 \cdot x.x^{1/2} \cdot \frac{2}{3}\right) - \frac{x^3}{12}\right]_0^4 = \left[\left(2.x.\sqrt{x}.\frac{2}{3}\right) - \frac{x^3}{12}\right]_0^4$$
$$= \left[\left(2.4.\sqrt{4} \cdot \frac{2}{3}\right) - \frac{4^3}{12}\right]_0^4 = \left[\left(2.4.2.\frac{2}{3}\right) - \frac{64}{12}\right]$$
$$= \frac{32}{3} - \frac{64}{12} = \frac{128 - 64}{12} = \frac{64}{12} = 5.33.$$

3. Calculate the area beneath the curve $y = x^3$ between $x = 3$ and $x = 6$.

Solution :

$$\text{The required area} = \int_3^6 x^3\, dx$$
$$= \left[\frac{x^{3+1}}{3+1}\right]_3^6 = \left[\frac{x^4}{4}\right]_3^6 = \left[\frac{6^4}{4}\right] - \left[\frac{3^4}{4}\right]$$
$$= \frac{1296}{4} - \frac{81}{4} = \frac{1296 - 81}{4} = \frac{1215}{4} = 303.75.$$

EXERCISE 7.3

1. Find the area of the portion bound between the curve $y^2 = 4ax$ and the X axis $x = 0$ and $x = 2a$
2. Find the area of the curve $y^2 = 2x$ and the line $y = 4x - 1$.
3. Find the area above X axis bound by $y = 4x - x^2 - 3$, $x = 1$ and $x = 3$.
4. Find the area between $y = x^3$ and $y = 9x^2$.
5. Find the area above the X axis bound by $x - 2y + 4 = 0$, $x = 3$ and $x = 6$.

APPLICATIONS IN ECONOMICS AND BUSINESS

COST FUNCTIONS

In differentiation, we have seen that if total cost (C) of producing an output x is given, we have to find out the Marginal Cost (MC), that is first order derivative of the total cost (C).

i.e., $MC = \frac{dC}{dx}$

Thus in integration, if MC is given, we have to find out the total cost (C). That is Total Cost (C) is the integral of MC, $\left(\text{i.e., } \frac{dC}{dx}\right)$ with reference to x.