CHAPTER 6

DIFFERENTIAL CALCULUS

CALCULUS

MEANING: Calculus is the branch of mathematics of change, motion and growth in related variables. It is the science of fluctuations. Therefore, in Economics, Calculus has a role to play when we consider how the sales is affected when there is change in the price or how the total cost, price etc., are affected when there is change in output and so on.

Branches of Calculus

The branches of Calculus are:

A) Differential Calculus and

B) Integral Calculus.

A) Differential Calculus

Meaning: Differentiation is the process of finding the rate at which a variable quantity is changing. To express the rate of change in any function, we have the concept of the 'Derivative' which involves small change in the dependent variables with reference to a small change in independent variables. The problem is to find a function derived from the given relationship between the two variables so as to express the idea of change. This derived function is called the "Derivative" of a given function. The process of obtaining the derivative is called "Differentiation". When a function has a Derivative, it is said to be Differentiable.

Thus,

1) The Marginal Cost is the rate of change of total cost with change in quantity produced.
2) The Marginal Revenue is the rate of change of total revenue with change in quantity produced.
3) The Marginal Utility is the rate of change of total utility with change in quantity consumed.
4) The Marginal Productivity is the rate of change of total productivity with change in factors of production.

A) Differential Calculus of One Variable

Let 'Y' be a continuous and single valued function of 'x'. Then an increase in the value of x will produce a corresponding increase or decrease in the value of Y. That is a function is differentiable at a point only if it is a single valued function of a continuous variable at that point. In other words, the derivative of y with respect to x exists only when the function is a single valued function of a continuous variable.

Therefore, $Y = f(x)$ — (1)

which is read as 'Y is a function of x'.

Give an increase in the value of x. This will produce a corresponding increase or decrease in the value of y. These increments are generally denoted by the symbols (Greek equivalent) Δx, Δy respectively.

Let the new value be,

$$Y + \Delta y = f(x + \Delta x) \quad \text{— (2)}$$

By equations (1) and (2)

$$\Delta Y = f(x + \Delta x) - Y$$

$$\text{or} \quad \Delta Y = f(x + \Delta x) - f(x) \quad \{\text{since } Y = f(x)\} \quad \text{— (3)}$$

For one unit increment in x, cause how much increment in y? That is the average rate of change of Y with respect to x is

$$\Delta x \text{ — } \Delta y$$

$$\therefore \quad 1 \text{ — } \frac{\Delta y}{\Delta x}$$

In other words, dividing both sides of the equation (3) by Δx

$$\frac{\Delta Y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \text{— (4)}$$

The average rate of change $\left(\frac{\Delta y}{\Delta x}\right)$ tends to a definite limit as Δx tends to zero or allow $\Delta x \to 0$,

$$\underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \text{— (5)}$$

If, in the right hand side, limit exists, then the left hand side limit written as $\frac{dy}{dx}$ and is called " Differential Co - efficient".

The "Differential Co - efficient" is also called the "Derivative".

This $\frac{dy}{dx}$ implies derivative of y with respect to x. The Differential Co-efficient of f(x) is written generally as $\frac{d}{dx}$ (fx) or $f'(x)$ or y' or $\frac{df}{dx}$ where $y = f(x)$.

$$\frac{dy}{dx} = f'(x) = \underset{\Delta x \longrightarrow 0}{\text{Limit}} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Thus, $\frac{dy}{dx}$ is the rate of change of y with respect to a change in x and is called the derivative of the function y with respect to x.

RULES OF DIFFERENTIATION

Rule No. I

Polynomial Functions Rule or Power Functions Rule

The derivative of a power function $y = f(x) = x^n$ (where 'n' is any real number) is nx^{n-1}.

Symbolically,

a) If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$

b) If $y = x$, then $\frac{dy}{dx} = x^{1-1} = x^0 = 1$ (since anything to the power of zero is 1)

Examples:

1. If $y = x^{10}$, $\frac{dy}{dx} = 10x^{10-1} = 10x^9$.
2. If $y = x^{-7}$, $\frac{dy}{dx} = -7x^{-7-1} = -7x^{-8} = \frac{-7}{x^8}$.
3. If $f(x) = x^{3/2}$, $f'(x) = \frac{3}{2}x^{3/2-1} = \frac{3}{2}x^{1/2}$.
4. If $y = x^{-8/3}$, $f'(x) = -\frac{8}{3}x^{-8/3-1} = \frac{-8}{3}x^{-11/3}$.
5. If $y = \sqrt{x}$, $y = x^{1/2}$, $\frac{dy}{dx} = 1/2\,x^{-1/2}$.
6. Determine the derivative of $f(x) = x^{5.7}$
   $\frac{dy}{dx} = 5.7x^{4.7}$.
7. If $y = \frac{1}{x^9}$, or $y = x^{-9}$, $\therefore \frac{dy}{dx} = -9x^{-10} = \frac{-9}{x^{10}}$

Rule No. II

Constant Function Rule

A) Derivative of a Constant

The Derivative of a Constant Function $\{y = f(x) = c\}$ is zero, i.e., constant disappears under differentiation. In other words, the increment dx in x will produce no change in 'c'.

If $y = C$, where C is Constant,

Then $\frac{dy}{dx} = \frac{d}{dx}(C) = 0$

Examples:

1. If $y = 7$, $\frac{dy}{dx} = 0$. 2. If $y = 115$, $\frac{dy}{dx} = 0$. 3. If $y = 73$, $\frac{dy}{dx} = 0$.

B) Derivative of the product of a constant and a function

The Derivative of the product of a constant and a function is equal to the product of that constant and the derivative of that function.

If $y = ax^n$, where 'a' is constant and $a \neq 0$,

then $\frac{dy}{dx} = \frac{d}{dx}(ax^n) = a\frac{d}{dx}(x^n)$

$$\frac{dy}{dx} = anx^{n-1}$$

Examples:

1. If $y = 10x^{12}$, find $\frac{dy}{dx}$.
   $\frac{dy}{dx} = 10\frac{d}{dx}x^{12}$
   $= 10(12x^{11})$
   $= 120x^{11}$.
2. If $y = 3x^4$, find $\frac{dy}{dx}$.
   $\frac{dy}{dx} = 3\frac{d}{dx}(x^4)$
   $= 3(4x^3)$
   $= 12x^3$.
3. If $y = 7x^5$, find $\frac{dy}{dx}$.
   $\frac{dy}{dx} = 7\frac{d}{dx}(x^5)$
   $= 7(5x^4)$
   $= 35x^4$.
4. If $y = 5x$, find $\frac{dy}{dx}$.
   $\frac{dy}{dx} = 5\frac{d}{dx}(x)$
   $= 5(1)$
   $= 5$.
5. If $y = 9x^{-4}$, find $\frac{dy}{dx}$.
   $\frac{dy}{dx} = -36x^{-5}$
6. If $y = -4x^7$, find $\frac{dy}{dx}$.
   $\frac{dy}{dx} = -28x^6$
7. If $y = -11x^{-9}$, find $\frac{dy}{dx}$.
   $\frac{dy}{dx} = 99x^{-10}$

Rule No. III

Linear Function Rule

If $y = mx + c$, where 'm' and 'c' are constants, then

$$\frac{dy}{dx} = \frac{d}{dx}(y) = \frac{d}{dx}(mx + c)$$

Examples:

1. If $y = 3x + 6$, find $\frac{dy}{dx}$

   $\frac{dy}{dx} = 3$

2. If $y = 9x + 2$, find $\frac{dy}{dx}$

   $\frac{dy}{dx} = 9$

3. If $y = 2x + 3$, find $\frac{dy}{dx}$

   $\frac{dy}{dx} = 2$

Rule No. IV

Addition Rule or Derivative of a Sum or Sum Rule

The Derivative of a Sum of two functions is simply equal to the sum of the separate derivatives.

If $y = u + v$, where u and v are the differentiable functions of x,

then $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ (i.e., Derivative of the first function + Derivative of the second function)

Examples:

1. If $y = x^3 + x^8$, find $\frac{dy}{dx}$.

   $\frac{dy}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}(x^8)$

   $= 3x^2 + 8x^7.$

2. If $y = x^4 + x^9 + x^{11}$, find $\frac{dy}{dx}$.

   $\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(x^9) + \frac{d}{dx}(x^{11})$

   $= 4x^3 + 9x^8 + 11x^{10}.$

3. If $y = 7x^5 + 3$, find $\frac{dy}{dx}$.

   $\frac{dy}{dx} = 7\frac{d}{dx}(x^5) + \frac{d}{dx}(3)$

   $= 7(5x^4) + 0$

   $= 35x^4.$

4. If $y = 2x^2 + 3x^8$, find $\frac{dy}{dx}$.

   $\frac{dy}{dx} = 2\frac{d}{dx}(x^2) + 3\frac{d}{dx}(x^8)$

   $= 2(2x) + 3(8x^7)$

   $= 4x + 24x^7.$

5. If $y = 11x^{-3} + 4x^{-9} + 3x + 7$, find $\frac{dy}{dx}$.

   $\frac{dy}{dx} = 11\frac{d}{dx}(x^{-3}) + 4\frac{d}{dx}(x^{-9}) + \frac{d}{dx}(3x) + \frac{d}{dx}(7)$

   $= -33x^{-4} - 36x^{-10} + 3.$

6. If $y = 7x^4 + x^9 + 6x + 11$, find $\frac{dy}{dx}$.

   $\frac{dy}{dx} = 7\frac{d}{dx}(x^4) + \frac{d}{dx}(x^9) + 6\frac{d}{dx}(x) + \frac{d}{dx}(11)$

   $= 7(4x^3) + 9x^8 + 6(1) + 0$

   $= 28x^3 + 9x^8 + 6$

Rule No. V

Subtraction Rule

The derivative of a difference of two functions is equal to the difference of the separate derivatives.

If $y = u - v$, where u and v are the differentiable functions of x.

then $\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$ (i.e., Derivative of the the first function – Derivative of the second function)

Examples:

1. If $y = x^3 - x^8$, find $\frac{dy}{dx}$.

   $\frac{dy}{dx} = \frac{d}{dx}(x^3) - \frac{d}{dx}(x^8)$

   $= 3x^2 - 8x^7.$

2. If $y = x^4 - x^9 - x^{11}$, find $\frac{dy}{dx}$.

   $\frac{dy}{dx} = \frac{d}{dx}(x^4) - \frac{d}{dx}(x^9) - \frac{d}{dx}(x^{11})$

   $= 4x^3 - 9x^8 - 11x^{10}.$

3. If $y = 7x^7 - 8x^8$, find $\frac{dy}{dx}$.

   $\frac{dy}{dx} = 7\frac{d}{dx}(x^7) - 8\frac{d}{dx}(x^8)$

   $= 7(7x^6) - 8(8x^7)$

   $= 49x^6 - 64x^7.$

4. If $y = 5x^7 - 8$, find $\frac{dy}{dx}$.

   $\frac{dy}{dx} = 5\frac{d}{dx}(x^7) - \frac{d}{dx}(8)$

   $= 5(7x^6) - 0$

   $= 35x^6.$

5. If $y = 10x^3 - x^4 - 7x - 21$, find $\frac{dy}{dx}$.

   $\frac{dy}{dx} = 10\frac{d}{dx}(x^3) - \frac{d}{dx}(x^4) - 7\frac{d}{dx}(x) - \frac{d}{dx}(21)$

   $= 10(3x^2) - 4x^3 - 7(1) - 0$

   $= 30x^2 - 4x^3 - 7.$

6. If $y = 3x^{-9} - 7x^{-7} - 8x - 8$, find $\frac{dy}{dx}$.

   $\frac{dy}{dx} = -27x^{-10} + 49x^{-8} - 8.$

Rule No. VI

Product Rule or Multiplication Rule or Derivative of a Product

The Derivative of the product of two functions is equal to the first function multiplied by the derivative of the second function plus the second function multiplied by the derivative of the first function.

If $y = uv$, where u and v are the differentiable functions of x, then $\frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$ [(i.e., (I function) (Derivative of the II function) + (II function) (Derivative of the I function)].

Examples:

1. [illegible]

Solution:

[illegible]

2. [illegible]

Solution:

[illegible]

3. Determine $\frac{dy}{dx}$, if $y = (x^4 + x^3)(x^2 + x)$

Solution:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = (x^4 + x^3)\frac{d}{dx}(x^2 + x) + (x^2 + x)\frac{d}{dx}(x^4 + x^3)$$

$$= (x^4 + x^3)(2x + 1) + (x^2 + x)(4x^3 + 3x^2)$$

$$= 2x^5 + x^4 + 2x^4 + x^3 + 4x^5 + 3x^4 + 4x^4 + 3x^3$$

$$= 6x^5 + 10x^4 + 4x^3$$

4. Evaluate $\frac{dy}{dx}$, if $y = (3x^2 + 2x)(x + 9)$.

Solution:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = (3x^2 + 2x)(1) + (x + 9)(6x + 2)$$

$$= 3x^2 + 2x + 6x^2 + 2x + 54x + 18$$

$$= 9x^2 + 58x + 18$$

5. [illegible]

Solution:

[illegible]

6. [illegible]

Solution:

[illegible]

Aliter:

[illegible]

7. If $y = x^3(3x^2 + 9)$, find $\frac{dy}{dx}$

Solution:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = x^3\frac{d}{dx}(3x^2 + 9) + (3x^2 + 9)\frac{d}{dx}(x^3)$$

$$= x^3(6x) + (3x^2 + 9)(3x^2)$$

$$= 6x^4 + 9x^4 + 27x^2 = 15x^4 + 27x^2.$$

Aliter:

$$y = x^3(3x^2 + 9) = 3x^5 + 9x^3$$

$$\frac{dy}{dx} = 15x^4 + 27x^2$$

Rule No. VII

Quotient Rule or Division Rule or Derivative of the Quotient of Two Functions

The Derivative of the Quotient of two functions is equal to the Denominator multiplied by the derivative of the Numerator minus the Numerator multiplied by the derivative of the Denominator, all divided by the square of the Denominator.

If $y = \frac{u}{v}$, where u and v are the differentiable functions of x and $v \neq 0$,

then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

i.e., $\frac{[(\text{Denominator})(\text{Derivative of Numerator})] - [(\text{Numerator})(\text{Derivative of Denominator})]}{(\text{Denominator})^2}$

Examples:

1. Evaluate $\frac{dy}{dx}$ for $y = \frac{x+1}{x-1}$.

Solution:

$$\frac{dy}{dx} = \frac{(x-1)\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(x-1)}{(x-1)^2}$$

$$= \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} = \frac{(x-1) - (x+1)}{(x-1)^2}$$

$$= \frac{x-1-x-1}{(x-1)^2} = \frac{-2}{(x-1)^2}.$$

2. Determine $\frac{dy}{dx}$ for $y = \frac{x-1}{x+1}$.

Solution:

$$\frac{dy}{dx} = \frac{(x+1)\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(x+1)}{(x+1)^2}$$

$$= \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2}$$

$$= \frac{(x+1) - (x-1)}{(x+1)^2} = \frac{x+1-x+1}{(x+1)^2} = \frac{2}{(x+1)^2}.$$

3. Find the derivative of the function $y = \frac{x^2+1}{x-1}$.

Solution:

$$\frac{dy}{dx} = \frac{(x-1)(2x) - (x^2+1)(1)}{(x-1)^2}$$

$$= \frac{2x^2 - 2x - (x^2+1)}{(x-1)^2} = \frac{2x^2 - 2x - x^2 - 1}{(x-1)^2} = \frac{x^2 - 2x - 1}{(x-1)^2}.$$

4. If $y = \frac{x-1}{x^2+1}$, find $\frac{dy}{dx}$.

Solution:

$$\frac{dy}{dx} = \frac{(x^2+1)(1) - (x-1)(2x)}{(x^2+1)^2}$$

$$= \frac{(x^2+1) - (2x^2 - 2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2+2x}{(x^2+1)^2} = \frac{-x^2+2x+1}{(x^2+1)^2}$$

5. If $y = \frac{x^2+1}{x+1}$, find $\frac{dy}{dx}$.

Solution:

$$\frac{dy}{dx} = \frac{(x+1)(2x) - (x^2+1)(1)}{(x+1)^2} = \frac{(2x^2+2x) - (x^2+1)}{(x+1)^2}$$

$$= \frac{2x^2+2x-x^2-1}{(x+1)^2} = \frac{x^2+2x-1}{(x+1)^2}.$$

6. If $y = \frac{x+1}{x^2+1}$, find $\frac{dy}{dx}$.

Solution:

$$\frac{dy}{dx} = \frac{(x^2+1)(1) - (x+1)(2x)}{(x^2+1)^2} = \frac{(x^2+1) - (2x^2+2x)}{(x^2+1)^2}$$

$$= \frac{x^2+1-2x^2-2x}{(x^2+1)^2} = \frac{1-2x-x^2}{(x^2+1)^2}.$$

7. If $y = \frac{x^2+1}{x^2-1}$, find $\frac{dy}{dx}$.

Solution:

$$\frac{dy}{dx} = \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2} = \frac{(2x^3-2x) - (2x^3+2x)}{(x^2-1)^2}$$

$$= \frac{2x^3-2x-2x^3-2x}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}.$$

8. If $y = \frac{x^2-1}{x^2+1}$, find $\frac{dy}{dx}$.

Solution:

$$\frac{dy}{dx} = \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2} = \frac{2x^3+2x-(2x^3-2x)}{(x^2+1)^2}$$

$$= \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}.$$

9. Derive $\frac{dy}{dx}$ for $y = \frac{x^3}{x-2}$.

Solution:

$$\frac{dy}{dx} = \frac{(x-2)(3x^2) - (x^3)(1)}{(x-2)^2} = \frac{3x^3 - 6x^2 - x^3}{(x-2)^2} = \frac{2x^3 - 6x^2}{(x-2)^2}.$$

10. Find out $\frac{dy}{dx}$ for $y = \frac{x-2}{x^3}$.

Solution:

$$\frac{dy}{dx} = \frac{(x^3)(1) - (x-2)(3x^2)}{(x^3)^2}$$

$$= \frac{x^3 - (3x^3 - 6x^2)}{x^6} = \frac{x^3 - 3x^3 + 6x^2}{x^6} = \frac{-2x^3 + 6x^2}{x^6}.$$

11. Given $y = \frac{x^2+1}{x^3+2x}$, find out $\frac{dy}{dx}$.

Solution:

$$\frac{dy}{dx} = \frac{(x^3+2x)(2x) - (x^2+1)(3x^2+2)}{(x^3+2x)^2}$$

$$= \frac{2x^4 + 4x^2 - (3x^4 + 2x^2 + 3x^2 + 2)}{(x^3+2x)^2}$$

$$= \frac{2x^4 + 4x^2 - 3x^4 - 2x^2 - 3x^2 - 2}{(x^3+2x)^2} = \frac{-x^4 - x^2 - 2}{(x^3+2x)^2}.$$

12. Calculate $\frac{dy}{dx}$ for $y = \frac{x^3+2x}{x^2+1}$.

Solution:

$$\frac{dy}{dx} = \frac{(x^2+1)(3x^2+2) - (x^3+2x)(2x)}{(x^2+1)^2}$$

$$= \frac{3x^4 + 2x^2 + 3x^2 + 2 - (2x^4 + 4x^2)}{(x^2+1)^2}$$

$$= \frac{3x^4 + 2x^2 + 3x^2 + 2 - 2x^4 - 4x^2}{(x^2+1)^2} = \frac{x^4 + x^2 + 2}{(x^2+1)^2}.$$

Rule No. VIII

Chain Rule or Function of a Function Rule or Derivative of a Composite Function

If y is the function of u where u is the function of x, then the derivative of y with respect to x (i.e., $\frac{dy}{dx}$) is equal to the product of the derivative of y with respect to u and the derivative of u with respect to x.

If $y = F(u)$, where $u = f(x)$,

then $\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$

Examples:

1. If $z = 7y + 3$, where $y = 5x^2$, find $\frac{dz}{dx}$.

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = 7(10x) = 70x.$$

2. If $z = 15y$ and $y = 2x$, find $\frac{dz}{dx}$.

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = 15(2) = 30.$$

3. Find the derivative of the function z with respect to x (i.e., $\frac{dz}{dx}$), if $y = 4x$ and $z = y^2$.

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = 2y\ (4) = 2(4x)\ (4) = 8x\ (4) = 32x.$$

4. If $y = x^2$ and $x = 1 + t^2$, find $\frac{dy}{dt}$.

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = 2x\ (2t) = 2\ (1 + t^2)\ (2t) \ldots \text{(since } x = 1 + t^2)$$

$$= (2 + 2t^2)(2t) = 4t + 4t^3.$$

5. If $z = 15y^2$, where $y = 2x^2$, find $\frac{dz}{dx}$.

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = 30y\ (4x) = 30(2x^2)\ (4x) \ldots\ldots \text{(since } y = 2x^2)$$

$$= (60x^2)(4x) = 240x^3.$$

6. Find the derivative of the function z with respect to x (i.e., $\frac{dz}{dx}$), if $y = x^2 + 3x$ and $z = y^2 + 1$.

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = 2y\ (2x+3) = 2(x^2 + 3x)\ (2x+3) \ldots \text{(since } y = x^2 + 3x)$$

$$= (2x^2 + 6x)(2x + 3) = 4x^3 + 6x^2 + 12x^2 + 18x$$

$$= 4x^3 + 18x^2 + 18x$$

7. If $z = y^4 + 3y^3$ and $y = 6x + 9$, find $\frac{dz}{dx}$.

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = (4y^3 + 9y^2)(6)$$

$$= 24y^3 + 54y^2 = 24\ (6x + 9)^3 + 54(6x + 9)^2 \ldots \text{(since } y = 6x + 9)$$

$$= 5184x^3 + 25272x^2 + 40824x + 21870.$$

8. If $m = n^3$ and $n = p^2 + 4$, find $\frac{dm}{dp}$.

$\frac{dm}{dn} = 3n^2, \; \frac{dn}{dp} = 2p$

$\frac{dm}{dp} = \frac{dm}{dn} \cdot \frac{dn}{dp} = 3n^2 (2p) = 6n^2p$

9. Find $\frac{du}{dx}$ by using the Chain Rule Technique for the function.

$u = f(x^2 + 3x)^2$

$u = (x^2 + 3x)^2$

Let $y = (x^2 + 3x)$

$u = y^2$ and $y = x^2 + 3x$

$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = 2y\,(2x + 3)$

$= 2(x^2 + 3x)\,(2x + 3)$ ..... (since $y = x^2 + 3x$)

$= (2x^2 + 6x)\,(2x + 3) = 4x^3 + 6x^2 + 12x^2 + 18x$

$= 4x^3 + 18x^2 + 18x.$

10. Find $\frac{dz}{dx}$, if $z = (x^3 + 3x)^2$.

We find $\frac{dz}{dx}$ by the Chain Rule

$z = (x^3 + 3x)^2$

Let $y = (x^3 + 3x)$

$\therefore \; z = y^2$

$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = 2y\,(3x^2 + 3) = 2(x^3 + 3x)\,(3x^2 + 3)$

$= (2x^3 + 6x)\,(3x^2 + 3) = 6x^5 + 6x^3 + 18x^3 + 18x$

$= 6x^5 + 24x^3 + 18x.$

11. If $y = (3x^2 - 4x)^5$, find $\frac{dy}{dx}$.

We find $\frac{dy}{dx}$ by the Chain Rule

$y = (3x^2 - 4x)^5$

Let $u = 3x^2 - 4x$

$y = u^5$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 5u^4\,(6x - 4)$

$= 5\,(3x^2 - 4x)^4\,(6x - 4)$

12. If C be the total cost, x be the total output and the Total Cost Function be $C = (2 + 3x^2)^2$, find the Marginal cost.

Solution:

We find the Mariginal cost $\left(\frac{dC}{dx}\right)$ by the Chain Rule.

$C = (2 + 3x^2)^2$

Let $y = (2 + 3x^2)$ and $C = y^2$

Therefore, Marginal cost $\left(\text{i.e., } \frac{dC}{dx}\right) = \frac{dC}{dy} \cdot \frac{dy}{dx}$

$= 2y\,(6x) = 2\,(2 + 3x^2)\,(6x)$

$= (4 + 6x^2)\,(6x) = 24x + 36x^3.$

Aliter:

Total cost (i.e.,C) $= (2 + 3x^2)^2$

$= 4 + 12x^2 + 9x^4$ ...... [$(a + b)^2$ formula]

Therefore, MC $\left(\text{i.e., } \frac{dC}{dx}\right) = 24x + 36x^3.$

Rule No. IX

Parametric Function Rule

If both x and y are the differentiable functions of 't', then the derivative of y with respect to x is obtained by dividing the derivative of y with respect to 't' by the derivative of x with respect to t (t is the parameter).

That is if $x = f(t)$ and $y = g(t)$ then

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

Example:

Find $\frac{dy}{dx}$, if $x = at^3$ and $y = 3at$.

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{3a}{3at^2} = \frac{1}{t^2}.$

Rule No. X

Implicit Function Rule

An equation $y = x^2 + 7x - 8$ directly expresses y in terms of x. Hence, this function is *"Explicit Function"*.

Again an equation $x^2y + y + 3x = 0$ does not directly express y in terms of x. Hence, this function is *"Implicit Function"*.

To find $\frac{dy}{dx}$ for the Implicit Function, every term in both sides of the function is to be differentiated with reference to x.

Examples:

Find $\frac{dy}{dx}$ of the following functions:

1. $2x - 3y = 6$.

$$\frac{d}{dx}(2x) - \frac{d}{dx}(3y) = \frac{d}{dx}(6)$$
$$2 - 3\frac{dy}{dx} = 0$$
$$-3\frac{dy}{dx} = -2$$
$$\frac{dy}{dx} = \frac{-2}{-3} = \frac{2}{3}.$$

2. $2x^2 + 3xy = 3$.

$$\frac{d}{dx}(2x^2) + \frac{d}{dx}(3xy) = \frac{d}{dx}(3)$$
$$4x + 3x\frac{dy}{dx} + 3y = 0 \quad \text{... (Product Rule)}$$
$$3x\frac{dy}{dx} = -4x - 3y$$
$$\frac{dy}{dx} = \frac{-4x - 3y}{3x}.$$

3. $xy = 6$.

$$x.\frac{d}{dx}(y) + y.\frac{d}{dx}(x) = \frac{d}{dx}(6) \quad \text{... (Product Rule)}$$
$$x\frac{dy}{dx} + y = 0$$
$$x\frac{dy}{dx} = -y$$
$$\frac{dy}{dx} = \frac{-y}{x}.$$

4. $x^2y = 6$.

$$x^2\frac{dy}{dx} + y.\,2x = 0 \quad \text{... (Product Rule)}$$
$$x^2\frac{dy}{dx} = -2xy$$
$$\frac{dy}{dx} = -\frac{2xy}{x^2} \text{ or } \frac{-2y}{x}.$$

5. $xy + y^2 = 4$.

$$x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0 \quad \text{... (Product Rule)}$$
$$(2y + x)\frac{dy}{dx} + y = 0$$
$$(2y + x)\frac{dy}{dx} = -y$$
$$\frac{dy}{dx} = \frac{-y}{2y + x} = -\frac{y}{(2y + x)}.$$

6. $x^2 + 2xy + y^2 = 4$.

$$2x + \frac{dy}{dx}(2x) + 2y + 2y\frac{dy}{dx} = 0 \quad \text{... (Product Rule)}$$
$$\frac{dy}{dx}(2x + 2y) = -2x - 2y$$
$$\frac{dy}{dx} = \frac{-2x - 2y}{2x + 2y} = \frac{-2(x + y)}{2(x + y)} = -1.$$

Rule No. XI

Inverse Function Rule or Derivative of an Inverse Function

The derivative of the Inverse Function $\left(\text{i.e., } \frac{dx}{dy}\right)$ is equal to the reciprocal of the derivative of the original function $\left(\text{i.e., } \frac{1}{dy/dx}\right)$.

If $x = f(y)$, we may find the Inverse Functions as, $y = f(x)$

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

provided both the functions are single-valued.

Examples:

1. Find the derivative of the Inverse Function $y = x^2$.

$$\frac{dy}{dx} = 2x$$
$$\text{then} \quad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{2x}.$$

2. Find the derivative of the Inverse Function $y = x^3 + 3x$.

$$\frac{dy}{dx} = 3x^2 + 3$$
$$\text{then} \quad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{3x^2 + 3}.$$

3. Find the derivative of the Inverse Function $y = x^2 - 2x$.

$$\frac{dy}{dx} = 2x - 2$$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

$$\frac{dx}{dy} = \frac{1}{2x - 2}.$$

4. Find the derivative of the Inverse Function $y = (x^2 + 1)(x^3 + 2)$.

$$\frac{dy}{dx} = (x^2 + 1)(3x^2) + (x^3 + 2)(2x)$$

$$= 3x^4 + 3x^2 + 2x^4 + 4x$$

$$= 5x^4 + 3x^2 + 4x$$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

$$\frac{dx}{dy} = \frac{1}{5x^4 + 3x^2 + 4x}.$$

5. Find the derivative of the Inverse Function $y = \frac{x + 1}{x - 1}$.

$$\frac{dy}{dx} = \frac{(x - 1)(1) - (x + 1)(1)}{(x - 1)^2} = \frac{x - 1 - x - 1}{(x - 1)^2} = \frac{-2}{(x - 1)^2}$$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

$$\frac{dy}{dx} = \frac{1}{\frac{-2}{(x - 1)^2}} = 1.\frac{(x - 1)^2}{-2} = \frac{(x - 1)^2}{-2}$$

Rule No. XII

Derivative of a Derivative or Derivative of Higher Orders or Higher Derivatives or Successive Differentiation

$\frac{dy}{dx}$ is the first order derivative of 'y' with respect to 'x'. The second order derivative is obtained by differentiating the first order derivative with respect to 'x' and is written as

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$ $\frac{d^2y}{dx^2}$ is also denoted by $f''(x)$ or f_{xx}.

Higher order derivatives or Derivatives of Higher Order can be defined similarly.

Examples :

1. If $y = 5x^4 + 2x^3$, find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

$$\frac{dy}{dx} = 20x^3 + 6x^2, \ldots \text{(First Order Derivative)}$$

$$\frac{d^2y}{dx^2} = 60x^2 + 12x \ldots \text{(Second Order Derivative)}$$

$$\frac{d^3y}{dx^3} = 120x + 12 \ldots \text{(Third Order Derivative)}.$$

2. Find the Third Derivative of y, if $y = 3x^6 + 2x^4 + x^2$.

$$\frac{dy}{dx} = 18x^5 + 8x^3 + 2x$$

$$\frac{d^2y}{dx^2} = 90x^4 + 24x^2 + 2$$

$$\frac{d^3y}{dx^3} = 360x^3 + 48x.$$

3. If $y = 4x^5 + 3x + 5$, find the First, Second and Third Derivatives.

$$\frac{dy}{dx} = 20x^4 + 3$$

$$\frac{d^2y}{dx^2} = 80x^3$$

$$\frac{d^3y}{dx^3} = 240x^2.$$

4. If $y = x^4 + x^2 + x$, find the Third Derivative of y.

$$\frac{dy}{dx} = 4x^3 + 2x+1$$

$$\frac{d^2y}{dx^2} = 12x^2 + 2$$

$$\frac{d^3y}{dx^3} = 24x.$$

Rule No. XIII

Exponential Function Rule

The derivative of an Exponential Function is the exponential itself, if the base of the function is 'e', the natural base of an exponential.

A) i) If $y = e^x$, then $\frac{dy}{dx} = \frac{d}{dx}(e^x) = e^x$

ii) If $y = e^{-x}$, then $\frac{dy}{dx} = -e^{-x}$

iii) If $y = e^{ax}$, then $\frac{dy}{dx} = ae^{ax}$ ... (a is constant)

B) If $y = e^u$, where $u = g(x)$, then

$\frac{dy}{dx} = \frac{d}{dx}(e^u) \cdot \frac{du}{dx} = e^u \left(\frac{du}{dx}\right)$

Examples :

1. Determine the derivative of the function $f(x) = 8e^x$.

$\frac{dy}{dx} = 8e^x$

2. If $y = e^{x^3+3}$, find $\frac{dy}{dx}$.

Let $u = x^3 + 3$

$y = e^u$

Hence, $\frac{dy}{dx} = e^u \left(\frac{du}{dx}\right) = e^u (3x^2) = 3x^2 (e^u) = 3x^2 \left(e^{x^3+3}\right)$ (since $u = x^3+3$).

3. If $y = e^{x^2}$, find $\frac{dy}{dx}$.

Let $u = x^2$

$y = e^u$

Hence, $\frac{dy}{dx} = e^u \left(\frac{du}{dx}\right) = e^u (2x) = e^{x^2} (2x)$ (Since $u = x^2$).

4. If $y = e^{2x^2}$, find the derivative.

Let $u = 2x^2$

$y = e^u$

Hence $\frac{dy}{dx} = e^u \left(\frac{du}{dx}\right) = e^u (4x) = 4x (e^u) = 4x (e^{2x^2})$ (since $u = 2x^2$).

5. Find $\frac{dy}{dx}$ for $y = e^{3x^3+5x^2+7}$

$y = e^{3x^3+5x^2+7}$

Let $u = 3x^3 + 5x^2 + 7$ and $y = e^u$

Hence, $\frac{dy}{dx} = e^u \left(\frac{du}{dx}\right)$

$= e^u (9x^2 + 10x) = (9x^2 + 10x) (e^u)$

$= (9x^2 + 10x) e^{3x^3+5x^2+7}$ (since $u = 3x^3 + 5x^2 + 7$).

Rule No. XIV

Logarithmic Function Rule

A) The derivative of a logarithm with natural base such as

$$y = \log x, \text{ then } \frac{dy}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$$

B) If $y = \log u$, where $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{du}(\log u) \cdot \frac{du}{dx}$

or $\frac{dy}{dx} = \frac{1}{u}\left(\frac{du}{dx}\right)$

Example:

If $y = \log x^5$, find $\frac{dy}{dx}$.

Let $u = x^5$, $y = \log u$, $\frac{dy}{dx} = \frac{1}{u}\left(\frac{du}{dx}\right) = \frac{1}{u}(5x^4) = \frac{5x^4}{u}$

$= \frac{5x^4}{x^5} = \frac{5}{x}$ (since $u = x^5$)

Aliter:

$y = \log x^5 = 5 \log x$

$\frac{dy}{dx} = 5\left(\frac{1}{x}\right) = \frac{5}{x}$

The Derivative of Trigonometric Functions

The following are the some standard results to find the derivatives of Trigonometric functions

Table 6.1 : Derivative of Trigonometric Functions

S.No	Function	Derivative $\left(\text{i.e., } \frac{dy}{dx}\right)$
1.	$y = \sin x$	$\cos x$
2.	$y = \cos x$	$-\sin x$
3.	$y = \tan x$	$\sec^2 x = \frac{1}{\cos^2 x}$
4.	$y = \cot x$	$-\mathrm{cosec}^2 x = -\frac{1}{\sin^2 x}$
5.	$y = \sec x$	$\sec x \tan x$
6.	$y = \mathrm{cosec}\, x$	$-\mathrm{cosec}\, x \cot x$

B) Differential Calculus of Two or More Variables

We shall discuss the Derivative of functions of two independent variables only, since the functions of more variables are analogous

II. Find partial derivatives for the following functions:

1. $u = xy$
2. $u = xy^2 + 3x + 2y$
3. $z = x^2y + xy^2$
4. $z = x^2y^2 + x^3 + y^3$

III. Find the total differentials for the following functions:

1. $u = x^2 + 2y^2$
2. $u = (x^2 + y^2)(2x^2 - y)$
3. $u = \frac{x^2 + y^2}{x - y}$
4. $u = \frac{x}{x^2 + y^2}$

MAXIMA AND MINIMA

A. Maxima

A function $f(x)$ is said to have attained its "Maximum value" or "Maxima" at $x = a$, if the function stops to increase and begins to decrease at $x = a$. In other words, $f(x_1)$ is a maximum value of a function 'f', if it is the highest of all its values for values of x in some neighbourhood of A (Fig 6.1).

B. Minima

A function $f(x)$ is said to have attained its "Minimum value" or "Minima" at $x = b$, if the function stops to decrease and begins to increase at $x = b$. In other words, $f(x_2)$ is a minimum value of a function 'f', if it is the lowest of all its values for values of x in some neighbourhood of B (Fig 6.1).

The Maxima and Minima of the function are called the "Extreme Values" of the function.

Maxima And Minima Of One Variable

Let us consider a function $Y = f(x)$. If we plot this function, the function takes the form as given in the Figure 6.1. We consider three points A, B, and C, where $\frac{dY}{dx} = 0$ in each case. That is in all stationary level the derivative is zero.

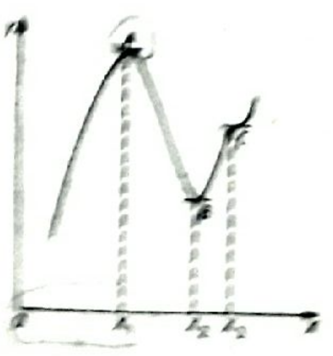

Fig 6.1 : Maxima and Minima

(1) At Point 'A'

We call the point A a maximum point because Y has a maximum value at this point when $X = OX_1$. Therefore, we say that at point A Y is maximum. It means that the value of Y at A is higher than the value on either side of A and also on the left side of point A the curve is increasing and decreasing on the right side of A. This also means that value of Y increases with the increase in X up to the point A. But it must fall after point A has been reached. As value of X increases from X_1 to X_2, the slope of the curve is decreasing or the slope of the curve changes from zero to negative values (since $\frac{dY}{dx} = 0$ at point A). Thus, if A is to be maximum,

i) $\frac{dY}{dx} = 0$ and also ii) $\frac{d^2Y}{dx^2} < 0$ i.e., -ve at point 'A'.

(2) At Point 'B'

We call the point 'B' a minimum point because Y has a minimum value when $X = OX_2$. Therefore, we say that at point 'B' Y is minimum. It means that the value of Y at B is lower than any value on either side of B and also on the left side of point B, the curve is decreasing and increasing on the right side of B. This also means that value of Y decreases with the increase in X upto the point B. But it must increase after point B has been reached. As value of X increases from X_2 to X_3, the slope of the curve is increasing or the slope of the curve changes from zero to positive values (since $\frac{dY}{dx} = 0$ at point B).

Thus if B is to be minimum,

i) $\frac{dY}{dx} = 0$ and also ii) $\frac{d^2Y}{dx^2} > 0$ i.e., +ve at point B.

Table 6.2 : Conditions for Maxima and Minima

		Maxima	Minima
1	First Order Condition (Necessary Condition)	$f'(x)$ or $\frac{dY}{dx} = 0$	$f'(x)$ or $\frac{dY}{dx} = 0$
2	Second Order Condition (Sufficient Condition)	$f''(x)$ or $\frac{d^2Y}{dx^2} < 0$ or -ve	$f''(x)$ or $\frac{d^2Y}{dx^2} > 0$ or +ve

(3) At Point 'C'

Point of Inflexion is a point at which a curve is changing from concave upward to concave downward, or vice versa. In the Figure 6.1 we call the point C a 'Point of Inflection' or 'Inflexional Point'. Because on either sides of point C the curve slopes upwards. Therefore, on either sides of C the first order derivative is greater than zero i.e., positive except at point C. Hence, this point is called 'Inflexional Point' or 'Point of Inflexion'.

because of mere bend in the curve. At the inflexional points, the second order derivative is equal to zero. "Inflexional points may be stationary and inflexional. In this case, both first and second order derivatives are zero. If the point is inflexional and non statioary, the first order derivative is not equal to zero. But the second order derivative is equal to zero.

Thus, if 'C' is to be "Inflexional point", then

i) $\frac{dY}{dx} \geq 0$ and also ii) $\frac{d^2Y}{dx^2} = 0$.

Examples:

1. Given the function $y = x^3 - 3x^2 + 7$, find the point of inflexion.

Solution:

$y = x^3 - 3x^2 + 7$

$\frac{dy}{dx} = 3x^2 - 6x$

The first condition for inflexion is $\frac{dy}{dx} = 0$

Therefore, $3x^2 - 6x = 0$

$x(3x - 6) = 0$

i) $x = 0$ or ii) $3x - 6 = 0$

$3x = 6$

$x = \frac{6}{3} = 2.$

a) If $x = 0$, $y = (0)^3 - 3(0)^2 + 7$

$y = 7$

b) If $x = 2$, $y = (2)^3 - 3(2)^2 + 7$

$y = 8 - 12 + 7 = 3$

Thus, $\frac{dy}{dx} = 0$, when $x = 0$ or $x = 2$.

The second condition for inflexion is $\frac{d^2y}{dx^2} = 0$

$\frac{d^2y}{dx^2} = 6x - 6$

$6x - 6 = 0$

$6x = 6$

$x = \frac{6}{6} = 1 > 0.$

The point of inflexion is $x = 0$ and $y = 7$ or $x = 2$ and $y = 3$.

2. Find the maxima or minima of the function $y = x^2 - 4x - 5$.

Solution:

$y = x^2 - 4x - 5$ $\qquad \frac{dy}{dx} = 2x - 4$

At the maximum or minimum $\frac{dy}{dx} = 0$

Therefore, $2x - 4 = 0$

$2x = 4$

$x = \frac{4}{2} = 2.$

The function may be a minimum or maximum at $x = 2$

$\frac{dy}{dx} = 2x - 4$

Therefore, $\frac{d^2y}{dx^2} = 2 > 0$ i.e., positive value.

Therefore, this function has minimum at $x = 2$

3. Find the maxima and minima of the function $y = 2x^3 - 6x$

Solution:

$y = 2x^3 - 6x$ $\qquad \frac{dy}{dx} = 6x^2 - 6$

At the maximum or minimum $\frac{dy}{dx} = 0$

Therefore, $6x^2 - 6 = 0$

$6x^2 = 6$

$x^2 = \frac{6}{6}$

$x^2 = 1$

$x = \pm 1$

$x = -1$ and $x = 1$ give maximum or minimum

$\frac{dy}{dx} = 6x^2 - 6$ $\qquad \frac{d^2y}{dx^2} = 12x$

When $x = -1$, $\frac{d^2y}{dx^2} = -12 < 0$ i.e., negative

When $x = 1$, $\frac{d^2y}{dx^2} = 12 > 0$ i.e., positive.

Therefore, $x = -1$ gives the maximum value of the function and $x = 1$ gives the minimum value of the function.

4. Find the maxima and minima of the following function $y = 2x^3 - 3x^2 - 36x + 10$.

Solution :

$y = 2x^3 - 3x^2 - 36x + 10$ $\qquad \frac{dy}{dx} = 6x^2 - 6x - 36$

At the maximum or minimum $\frac{dy}{dx} = 0$

Therefore, $6x^2 - 6x - 36 = 0$

$x^2 - x - 6 = 0$

$x^2 - 3x + 2x - 6 = 0$

$x(x - 3) + 2(x - 3) = 0$

Therefore, $(x - 3)(x + 2) = 0$

Therefore, $x = 3$ or $x = -2$

$-b \pm \sqrt{b^2 - 4ac}$
$2a$

$x = -2$, and $x = 3$ give maximum or minimum

$\frac{dy}{dx} = 6x^2 - 6x - 36 \qquad \frac{d^2y}{dx^2} = 12x - 6$

When $x = -2$, $\frac{d^2y}{dx^2} = -30 < 0$ i.e. negative

When $x = 3$, $\frac{d^2y}{dx^2} = 30 > 0$ i.e., positive

Therefore, $x = -2$ gives the maximum value of the function and

$x = 3$ gives the minimum value of the function.

5. **Investigate the maxima and minima of the function**
   $y = 3x^4 - 10x^3 + 6x^2 + 5$.

Solution :

$y = 3x^4 - 10x^3 + 6x^2 + 5 \qquad \frac{dy}{dx} = 12x^3 - 30x^2 + 12x$

At the maximum or minimum $\frac{dy}{dx} = 0$

Therefore,

$$
\begin{aligned}
12x^3 - 30x^2 + 12x &= 0\\
2x^3 - 5x^2 + 2x &= 0\\
x(2x^2 - 5x + 2) &= 0\\
x(2x^2 - 4x - x + 2) &= 0\\
x\{2x(x-2) - 1(x-2)\} &= 0\\
(x)(x-2)(2x-1) &= 0
\end{aligned}
$$

$x = 0$ or $x = 2$ or $2x = 1$

$x = \frac{1}{2}$

$x = 0$, $x = 2$ and $x = \frac{1}{2}$ give maximum or minimum

$\frac{dy}{dx} = 12x^3 - 30x^2 + 12x \qquad \frac{d^2y}{dx^2} = 36x^2 - 60x + 12$

When $x = 0$, $\frac{d^2y}{dx^2} = 12 > 0$ i.e., positive.

When $x = 2$, $\frac{d^2y}{dx^2} = 36 > 0$ i.e., positive.

When $x = \frac{1}{2}$ $\frac{d^2y}{dx^2} = -9 < 0$ i.e., negative.

Therefore, $x = 0$ gives the minimum value of the function,

$x = 2$ gives the minimum value of the function and

$x = \frac{1}{2}$ gives the maximum value of the function.

6. Determine the maxima and minima of the function
   $y = x^3 + 5x^2 + 8x + 5$

Solution:

$y = x^3 + 5x^2 + 8x + 5 \qquad \frac{dy}{dx} = 3x^2 + 10x + 8$

At the maximum and minimum $\frac{dy}{dx} = 0$

$$
\begin{aligned}
3x^2 + 10x + 8 &= 0\\
3x^2 + 6x + 4x + 8 &= 0\\
3x(x+2) + 4(x+2) &= 0\\
(x+2)(3x+4) &= 0
\end{aligned}
$$

i) $x = -2$

ii) $3x = -4$

$x = -\frac{4}{3}$

$x = -2$ and $x = -\frac{4}{3}$ give the maximum or minimum

$\frac{dy}{dx} = 3x^2 + 10x + 8 \qquad \frac{d^2y}{dx^2} = 6x + 10$

When $x = -2$, $\frac{d^2y}{dx^2} = 6(-2) + 10 = -12 + 10 = -2 < 0$ is –ve

When $x = -\frac{4}{3}$, $\frac{d^2y}{dx^2} = 6\left(-\frac{4}{3}\right) + 10 = -8 + 10 = 2 > 0$ is +ve

Therefore, $x = -2$ gives the maximum value of the function and

$x = -\frac{4}{3}$ gives the minimum value of the function.

7. Find the exterme values of the function
   $y = 15x^3 - 9x^2 - 8x$.

Solution:

$y = 15x^3 - 9x^2 - 8x \qquad \frac{dy}{dx} = 45x^2 - 18x - 8$

At the maximum and minimum $\frac{dy}{dx} = 0$

Therefore,

$$
\begin{aligned}
45x^2 - 18x - 8 &= 0\\
45x^2 - 30x + 12x - 8 &= 0\\
15x(3x-2) + 4(3x-2) &= 0\\
(3x-2)(15x+4) &= 0
\end{aligned}
$$

i) $3x - 2 = 0$ or ii) $15x + 4 = 0$

$3x = 2 \qquad 15x = -4$

$x = \frac{2}{3} \qquad x = -\frac{4}{15}$

$x = \frac{2}{3}$ and $x = -\frac{4}{15}$ give the maximum or minimum

$\frac{dy}{dx} = 45x^2 - 18x - 8$ $\frac{d^2y}{dx^2} = 90x - 18$

When $x = \frac{2}{3}$, $\frac{d^2y}{dx^2} = 90\left(\frac{2}{3}\right) - 18 = 60 - 18 = 42 > 0$ i.e., +ve.

When $x = -\frac{4}{15}$, $\frac{d^2y}{dx^2} = 90\left(-\frac{4}{15}\right) - 18 = -24 - 18 = -42 < 0$ i.e., −ve

Therefore, $x = \frac{2}{3}$ gives the minimum value of the function and

$x = -\frac{4}{15}$ gives the maximum value of the fucntion.

MAXIMA AND MINIMA OF TWO OR MORE VARIABLES

We shall dicuss the functions of two variables only, since the functions of more than two variables are analogous.

Maxima and Minima of Two Variables

Let us consider a function $U = f(x,y)$. If we plot this function, the function takes the form as given in the Figures 6.2 and 6.3.

Fig. 6.2 : Maxima

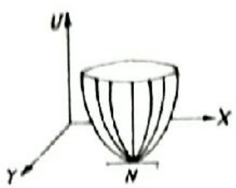

Fig. 6.3 : Minima

We call the point 'M' in Figure 6.2 a maximum, because the value of 'U' at this point is higher than any other value on either side of 'M'. Similarly, we call the point 'N' in Figure 6.3 as minimum, because the value of 'U' at this point is smaller than any other value on either side of 'N'.

If $U = f(x,y)$, then the conditions for maxima and minima are given in Table 6.3.

Table 6.3 : Conditions for Maxima and Minima

Maxima		Minima	
Alternative notation	f Notation	Alternative Notation	f Notation
1. $\frac{\partial u}{\partial x} = 0$	1. $f_x = 0$	1. $\frac{\partial u}{\partial x} = 0$	1. $f_x = 0$
2. $\frac{\partial u}{\partial y} = 0$	2. $f_y = 0$	2. $\frac{\partial u}{\partial y} = 0$	2. $f_y = 0$
3. $\frac{\partial^2 u}{\partial x^2} < 0$	3. $f_{xx} < 0$	3. $\frac{\partial^2 u}{\partial x^2} > 0$	3. $f_{xx} > 0$
4. $\frac{\partial^2 u}{\partial y^2} < 0$	4. $f_{yy} < 0$	4. $\frac{\partial^2 u}{\partial y^2} > 0$	4. $f_{yy} > 0$
5. $\frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 > 0$	5. $f_{xx} f_{yy} - (f_{xy})^2 > 0$	5. $\frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 > 0$	5. $f_{xx}f_{yy} - (f_{xy})^2 > 0$

If $U = f(x,y)$, then the conditions for 'saddle point' and 'no information' are given in Table 6.4.

Table 6.4 : Conditions for Saddle Point and No Information

Saddle point		No Information	
Alternative Notation	F Notation	Alternative Notation	F – Notation
1. $\frac{\partial u}{\partial x} = 0$	1. $f_x = 0$	1. $\frac{\partial u}{\partial x} = 0$	1. $f_x = 0$
2. $\frac{\partial u}{\partial y} = 0$	2. $fy = 0$	2. $\frac{\partial u}{\partial y} = 0$	2. $f_y = 0$
3. $\frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 < 0$	3. $f_{xx}f_{yy} - (f_{xy})^2 < 0$	3. $\frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2} \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = 0$	3. $f_{xx}f_{yy} - (f_{xy})^2 = 0$

Examples:

1. Investigate the maximum or minimum value of the following function
   $z = 48 - 4x^2 - 2y^2 + 16x + 12y.$

Solution:

$z = 48 - 4x^2 - 2y^2 + 16x + 12y$

$\frac{\partial z}{\partial x} = -8x + 16$

Since $\frac{\partial z}{\partial x} = 0$,

$$-8x + 16 = 0$$
$$-8x = -16$$
$$x = \frac{-16}{-8} = 2$$

$$\frac{\partial z}{\partial y} = -4y + 12$$

Since $\frac{\partial z}{\partial y} = 0$, $-4y + 12 = 0$

$$-4y = -12$$
$$y = \frac{-12}{-4} = 3$$

We have maximum or minimum value at (2,3)

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(-8x + 16) = -8 < 0$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(-4y + 12) = -4 < 0$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(-8x + 16) = 0$$

$$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = (-8 \times -4) - (0)^2 = 32 - 0 = 32 > 0$$

This function has minimum value at point (2, 3).

Minimum value

$$(2,3) = 48 - 4x^2 - 2y^2 + 16x + 12y$$
$$= 48 - 4(2)^2 - 2(3)^2 + 16(2) + 12(3)$$
$$= 48 - 16 - 18 + 32 + 36$$
$$= 116 - 34 = 82$$

2. Find the maxima or minima for the function
$z = 10x + 20y - x^2 - y^2$.

Solution:

$$z = 10x + 20y - x^2 - y^2$$
$$\frac{\partial z}{\partial x} = 10 - 2x$$
$$\frac{\partial z}{\partial y} = 20 - 2y$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(10 - 2x) = -2 < 0$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(20 - 2y) = -2 < 0$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(10 - 2x) = 0$$

i) $\frac{\partial z}{\partial x} = 0$

$$10 - 2x = 0$$
$$-2x = -10$$
$$x = 5$$

ii) $\frac{\partial z}{\partial y} = 0$

$$20 - 2y = 0$$
$$-2y = -20$$
$$y = 10$$

We get the point (5,10)

$$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = (-2)(-2) - (0)^2 = 4 - 0 = 4 > 0$$

Thus, z is maximum at (5, 10) and the maximum value
$= 10(5) + 20(10) - (5)^2 - (10)^2 = 50 + 200 - 25 - 100 = 125.$

3. Examine the maxima or minima of the function
$z = 3x^2 + y^2 - 3xy$.

Solution:

$$z = 3x^2 + y^2 - 3xy$$
$$\frac{\partial z}{\partial x} = 6x - 3y \quad \text{— (1)}$$
$$\frac{\partial z}{\partial y} = 2y - 3x \quad \text{— (2)}$$
$$\frac{\partial^2 z}{\partial x^2} = 6 > 0$$
$$\frac{\partial^2 z}{\partial y^2} = 2 > 0$$
$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(6x - 3y) = -3 < 0$$
$$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 6 \times 2 - (-3)^2$$
$$= 12 - 9$$
$$= 3 > 0$$

$6x - 3y = 0$ — (1)

$-3x + 2y = 0$ — (2)

$(1) \times -3 \quad = -18x + 9y = 0$ — (3)

$(2) \times 6 \quad = -18x + 12y = 0$ — (4)

$$-3y = 0$$
$$y = 0$$

Let us substituting the value of $y = 0$ in the equation (2)

$$2y - 3x = 0$$
$$2(0) - 3x = 0$$
$$-3x = 0$$
$$x = 0$$

We get the point (0,0).

$$\frac{\partial^2 z}{\partial x^2} = 6 > 0$$

$$\frac{\partial^2 z}{\partial y^2} = 2 > 0$$

Thus, z is minimum value at (0,0)
Minimum value = $3(0)^2 + (0)^2 - 3(0)(0) = 0$.

4. Find the extreme values of the function
$z = -x^2 + xy - y^2 + 2x + y$

Solution:

$$z = -x^2 + xy - y^2 + 2x + y$$

$$\frac{\partial z}{\partial x} = -2x + y + 2$$

$$\frac{\partial z}{\partial y} = x - 2y + 1$$

$$\frac{\partial^2 z}{\partial x^2} = -2 < 0 \text{ i.e., negative}$$

$$\frac{\partial^2 z}{\partial y^2} = -2 < 0 \text{ i.e., negative}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}(-2x + y + 2) = 1 > 0 \text{ i.e., positive}$$

The first order derivative $\frac{dz}{dx}$ and $\frac{dz}{dy}$ become zero.

Therefore,

$$-2x + y + 2 = 0 \quad \text{— (1)}$$
$$x - 2y + 1 = 0 \quad \text{— (2)}$$

Solve the equations (1) and (2)

$$(1) \times 1 \rightarrow \quad -2x + y + 2 = 0 \quad \text{— (3)}$$
$$(2) \times -2 \rightarrow \quad \underline{-2x + 4y - 2 = 0} \quad \text{— (4)}$$
$$(3) - (4) \rightarrow \quad -3y + 4 = 0$$
$$-3y = -4$$
$$3y = 4$$
$$y = \frac{4}{3}$$

Let us substitute the value of $y = \frac{4}{3}$ in the equation (1)

$$-2x + y + 2 = 0$$
$$-2x + \frac{4}{3} + 2 = 0$$
$$-2x = \frac{-4}{3} - 2$$
$$-2x = \frac{-4-6}{3} = \frac{-10}{3}$$
$$-x = \frac{-10}{3 \times 2} = \frac{-10}{6} = \frac{-5}{3}$$
$$x = \frac{5}{3}$$

We get the point $\left(\frac{5}{3}, \frac{4}{3}\right)$

$$\frac{d^2 z}{dx^2} \; -2 < 0 \qquad \frac{\partial^2 z}{\partial y^2} = -2 < 0$$

Thus, z has maximum at $\left(\frac{5}{3}, \frac{4}{3}\right)$ and the maximum value is,

$$z = -x^2 + xy - y^2 + 2x + y$$
$$= -\left(\frac{5}{3}\right)^2 + \left(\frac{5}{3} \times \frac{4}{3}\right) - \left(\frac{4}{3}\right)^2 + \left(2 \times \frac{5}{3}\right) + \frac{4}{3}$$
$$= \frac{-25}{9} + \frac{20}{9} - \frac{16}{9} + \frac{10}{3} + \frac{4}{3}$$
$$= \frac{-25 + 20 - 16 + 30 + 12}{9}$$
$$= \frac{62 - 41}{9} = \frac{21}{9} = 2.33.$$

5. Find the extreme values for the function
$z = 4x^2 - xy + y^2 - x^3$.

Solution:

$$z = 4x^2 - xy + y^2 - x^3$$

$$\frac{\partial z}{\partial x} = 8x - y - 3x^2 \qquad \frac{\partial z}{\partial y} = -x + 2y$$

The first order derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ become zero.

Therefore,

$$8x - y - 3x^2 = 0 \quad \text{— (1)}$$
$$-x + 2y = 0 \quad \text{— (2)}$$
$$(1) \times 2 \rightarrow 16x - 2y - 6x^2 = 0 \quad \text{— (3)}$$
$$(2) \times -1 \rightarrow +x - 2y = 0 \quad \text{— (4)}$$