CHAPTER 9

MATRICES

MEANING

In 1858, Cayley, the English Mathematician, invented the Theory of Matrices. An array of numbers (real or complex) in rectangular brackets is called 'Matrix'. In other words, Matrix is a collection of vectors. Each Row and Column of the Matrix is a Vector.

In Matrix, numbers are written in Square or Rectangular Brackets [ ] or Parentheses ( ) or pair double bars ‖ ‖.

NOTATIONS

This is a convention to denote Matrices by capital letters such as A, B, C, ... X, Y, Z. The numbers which form a Matrix or within a Matrix are called its "Elements". In other words, the entries in a Matrix are called "Elements". Usually elements are denoted by small letters a, b, c. . . .... x,y,z.

$$\text{Matrix } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}_{4 \times 4}$$

BASIC CONCEPTS

$$\begin{bmatrix} 5_{11} & 6 & 8 \\ 7 & 0_{22} & 9 \\ 3 & 8 & 0_{33} \end{bmatrix}$$

The numbers in horizontal lines are called the "Rows" of the Matrix (i.e., 5, 6, 8 is the I Row) and the numbers in vertical lines are called "Columns" of the Matrix (i.e, 5, 7 and 3 is the I Column). The rows are numbered from top to bottom whereas the columns are numbered from left to right. The main or principal diagonal of a matrix is made up of all the elements whose row position equals their column position.

In the above example, the elements 5,0, 0 are called as the "Diagonal Elements" and the other elements are "Off - Diagonal Elements" or "Non-Diagonal Elements." Elements above the Diagonal Elements (6, 8, and 9) are called "Upper Diagonal Elements" or "Super Diagonal Elements" and Elements lower the Diagonal Elements (7, 3 and 8) are called "Lower Diagonal Elements" or "Sub Diagonal Elements".

Order of the Matrix

The number of the rows and columns of a Matrix is called the "Order" or "Dimension" or "Size" of a Matrix.

By convention, the number of rows is given before the number of columns when discussing the 'Order'.

If a Matrix contains 'm' rows and 'n' columns, it is called a Matrix of order m × n and can be denoted in short as $[a_{ij}]_{m \times n}$

Matrix in the above example is a 3 × 3 (read 3 by 3) Matrix.

Examples :

Determine the order of the following Matrices.

1. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is a 2 × 3 Matrix or a Matrix of dimensions 2 × 3.

2. $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is a 3 × 2 Matrix. 3. $C = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is a 4 × 1 Matrix.

4. D = [1 2 3 4] is a 1 × 4 Matrix. 5. $E = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is a 3 × 3 Matrix.

TYPES OF MATRICES

1) Row Matrix

A Matrix of order (1 × n) having only one row and n columns is called a "Row Vector" or "Row Matrix". Row of the Martrix is a "Vector".

Examples :

1. $[a_{11}\ a_{12}\ a_{13}\ a_{14}]_{1\times4}$ 2. $[1\ \ 2\ \ 3]_{1\times3}$

2) Column Matrix

A Matrix of order (m × 1) having only one column and m rows is called a "Column Matrix" or "Column Vector". Column of the Matrix is a "Vector".

Examples:

1) $\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix}_{4 \times 1}$ 2) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3 \times 1}$

3) Zero Matrix or Null Matrix

If all the elements of a Matrix are zero, the Matrix is called a "Null Matrix" or "Zero Matrix". Null Matrix is denoted by the symbol φ or 0

Example

[illegible] Row Matrix of order 1 ... Row Matrix of order 2

[illegible]

[illegible] Sub-Matrix

[illegible]

Example

[illegible]

Equality of two Matrices or Equal Matrices

Two Matrices A and B are equal (A = B), if and only if they have the same order and their corresponding elements are equal. [illegible]

Example

1. If A = [illegible] and B = [illegible] then A = B.

2. If A = [illegible] and B = [illegible] then A ≠ B,

since corresponding elements are not equal.

Rectangular Matrix

If the number of rows of a Matrix is greater than the number of its columns or the number of columns of a Matrix is greater than the number of its rows, then the Matrix is called a "Rectangular Matrix", that is, m > n or n > m.

Example

[illegible]

Square Matrix

If the number of rows of a Matrix is equal to the number of its columns, then the Matrix is called a "Square Matrix". In other words, a Matrix is square if it has the same number of rows and columns.

That is, m = n or number of rows = number of columns.

Example

1. [illegible] is a 2 × 2 Square Matrix.

2. [illegible] is a Square Matrix of order 3.

3. [illegible] is a 4 × 4 Square Matrix.

Identity or Unit Matrix

A Square Matrix in which all the diagonal elements are one and all the non-diagonal elements are 0, is called "an Identity Matrix" or "an Unit Matrix". In other words, if in the Diagonal Matrix [illegible] then the Matrix is said to be an "Identity Matrix". Identity Matrix is denoted by the symbol "I".

Example

1. [1] Unit Matrix of order 1.

2. [illegible] is Unit Matrix of order 2.

3. [illegible] is Unit Matrix of order 3.

Diagonal Matrix

A Square Matrix in which all the non-diagonal elements are zero, is called a "Diagonal Matrix". In other words, a "Diagonal Matrix" is a Square Matrix with zeros except on the leading diagonal. That is, the Matrix which is both Upper and Lower Triangular Matrices is said to be a "Diagonal Matrix".

Examples :

1. $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ 2. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

7–C) Scalar Matrix

A Diagonal Matrix in which all the diagonal elements are equal is called a "Scalar Matrix". In other words, a Scalar Matrix is a Diagonal Matrix in which all the diagonal elements are equal. That is, if in the Diagonal Matrix $a_{11} = a_{22} = a_{33} \ldots = a_{nn} = K$ (Scalar), then the Matrix is said to be a "Scalar Matrix".

Examples :

1. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 2. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Unit Matrix is a Scalar Matrix. Likewise Scalar Matrix is a Diagonal Matrix. But Diagonal Matrix is not a Scalar Matrix.

7–D) Symmetric Matrix

A Matrix is said to be a "Symmetric Matrix", if it is a Square Matrix and $a_{ij} = a_{ji}$. Hence, if Matrix $A^T = A$, then the Matrix A is a "Symmetric Matrix". Therefore, Identity, Diagonal and Scalar Matrices are Symmetric Matrices.

Examples :

[illegible]

7–E) Skew Symmetric Matrix

[illegible]

Examples :

[illegible]

7–F) Idempotent Matrix

[illegible]

Examples :

1. A = [1], because $A \times A = A$ i) $[1] \times [1] = [1]$ ii) $A = 1$, $A^T = 1$. Therefore, $A^T = A$.

2. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, because (i) $A \times A = A$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

ii) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Therefore, $A^T = A$.

7–G) Nilpotent Matrix

A Nilpotent Matrix is a Symmetric Matrix for which $A^n = 0$ where 'n' is a positive integer.

7–H) Triangular Matrix

A Triangular Matrix is a Square Matrix in which all the elements above or below the principal diagonal are Zero.

a) Upper Triangular Matrix

An Upper Triangular Matrix is a Triangular Matrix in which all the elements below the principal diagonal (i.e., Lower Diagonal or Sub Diagonal elements) are zero. In other words, Upper Triangular Matrix is a Square Matrix whose elements [illegible]

Examples :

[illegible]

b) Lower Triangular Matrix

[illegible]

Examples :

[illegible]

7–I) Singular Matrix

[illegible]

Example :

$$A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} \qquad |A| = (3 \times 4) - (2 \times 6) = 12 - 12 = 0$$

7–J) Non Singular Matrix

A Square Matrix 'A' is said to be a "Non Singular Matrix", if its determinant is not equal to zero i.e. $|A| \neq 0$.

Example :

$$A = \begin{bmatrix} 1 & 7 \\ 3 & 1 \end{bmatrix} \qquad |A| = (1 \times 1) - (3 \times 7) = 1 - 21 = -20$$

7–K) Hermitian Matrix

A Square Matrix $A = [a_{ij}]$ is said to be a "Hermitian Matrix", if $A^T = A$. Thus, Matrix A is a Hermitian Matrix provided $a_{ij} = \bar{a}_{ji}$ for all i, j.

7–L) Skew-Hermitian Matrix

A Square Matrix $A = [a_{ij}]$ is said to be a "Skew Hermitian Matrix", if $A^T = -A$. Thus, Matrix 'A' is a Skew Hermitian Matrix provided $a_{ij} = -\bar{a}_{ji}$ for all i, j.

7–M) Orthogonal Matrix

An m × m Square Matrix 'A' is called an "Orthogonal Matrix", if $A^T A = AA^T = I$, where I is an m × m Unit Matrix.

7–N) Commutative or Commute

If A and B are Square Matrices and AB = BA, then A and B are called 'Commutative' or said to be 'Commute'.

7–O) Anti - Commute

If A and B are Square Matrices and AB = – BA, then these Matrices A and B are said to be "Anti-Commute".

8) Nullity of a Matrix

For a system of homogeneous equations AX = 0, the solution vectors X constitute a Vector space called the "Null space" of A. The dimension of this Null space is called the "Nullity of a Matrix" A and is denoted by N_A.

ALGEBRA OF MATRICES OR OPERATIONS WITH MATRICES

(A) ADDITION OR SUM

Two Matrices A and B can be added if and only if they have the same order i.e., the same number of rows and columns. That is number of column of Matrix A is equal to the number of column of B and number of Row of Matrix A is equal to the number of Row of B.

That is, Two Matrices of the same order are said to be "Conformable" for Addition. The sum of two Matrices of the same order is obtained by adding together corresponding elements of the two matrices. If $A = [a_{ij}]$ and $B = [b_{ij}]$, then C = A + B is the matrix having a general element of the form $C_{ij} = a_{ij} + b_{ij}$.

Examples :

1. If $A = \begin{bmatrix} 2 & 0 \\ -5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 6 \\ 4 & 1 \end{bmatrix}$, find A + B.

Solution :

$$A + B = \begin{bmatrix} 2 & 0 \\ -5 & 6 \end{bmatrix} + \begin{bmatrix} -3 & 6 \\ 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (2-3) & (0+6) \\ (-5+4) & (6+1) \end{bmatrix} = \begin{bmatrix} -1 & 6 \\ -1 & 7 \end{bmatrix}_{2 \times 2}.$$

2. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \\ 3 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 & 4 \\ 6 & 2 & 0 \\ 2 & 1 & 3 \end{bmatrix}$, find A + B.

Solution :

$$A + B = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \\ 3 & 0 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 3 & 4 \\ 6 & 2 & 0 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (1-1) & (2+3) & (-3+4) \\ (0+6) & (-1+2) & (2+0) \\ (3+2) & (0+1) & (4+3) \end{bmatrix} = \begin{bmatrix} 0 & 5 & 1 \\ 6 & 1 & 2 \\ 5 & 1 & 7 \end{bmatrix}_{3 \times 3}.$$

3. If $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \\ 8 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 10 & 2 \\ 8 & 6 \end{bmatrix}$, find A + B.

Solution :

A + B is not defined, since Dimensions are not equal.

4. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \end{bmatrix}$ and $C = \begin{bmatrix} 19 & 20 & 21 \\ 22 & 23 & 24 \\ 25 & 26 & 27 \end{bmatrix}$,

find A + B + C.

Solution

[illegible]

5. If A = [illegible], B = [illegible] and C = [illegible], find A + B + C

Solution

A + B + C is not defined, since the matrices have different orders.

6. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \\ 7 & 8 & -9 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -3 & 2 \\ 1 & 6 & -4 \\ -7 & 1 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 6 & 2 & 1 \\ 4 & 0 & 7 \\ 2 & 1 & 6 \end{bmatrix}$,

show that (i) A + B = B + A

(ii) A + (B + C) = (A + B) + C

Solution :

(i) A + B = B + A

$$A + B = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \\ 7 & 8 & -9 \end{bmatrix} + \begin{bmatrix} 4 & -3 & 2 \\ 1 & 6 & -4 \\ -7 & 1 & 3 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 5 & -1 & -1 \\ 5 & 1 & 2 \\ 0 & 9 & -6 \end{bmatrix}$$

$$B + A = \begin{bmatrix} 4 & -3 & 2 \\ 1 & 6 & -4 \\ -7 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \\ 7 & 8 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & -1 \\ 5 & 1 & 2 \\ 0 & 9 & -6 \end{bmatrix}$$ Therefore, A + B = B + A.

(ii) A + (B + C) = (A + B) + C

$$B + C = \begin{bmatrix} 4 & -3 & 2 \\ 1 & 6 & -4 \\ -7 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 6 & 2 & 1 \\ 4 & 0 & 7 \\ 2 & 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -1 & 3 \\ 5 & 6 & 3 \\ -5 & 2 & 9 \end{bmatrix}$$

$$A + (B + C) = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \\ 7 & 8 & -9 \end{bmatrix} + \begin{bmatrix} 10 & -1 & 3 \\ 5 & 6 & 3 \\ -5 & 2 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 1 & 0 \\ 9 & 1 & 9 \\ 2 & 10 & 0 \end{bmatrix}$$

$$(A + B) + C = \begin{bmatrix} 5 & -1 & -1 \\ 5 & 1 & 2 \\ 0 & 9 & -6 \end{bmatrix} + \begin{bmatrix} 6 & 2 & 1 \\ 4 & 0 & 7 \\ 2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 11 & 1 & 0 \\ 9 & 1 & 9 \\ 2 & 10 & 0 \end{bmatrix}$$

Therefore, A + (B + C) = (A + B) + C

LAWS OR PROPERTIES OF MATRIX ADDITION

In Matrix also Associative law and Commutative law are applicable.

a) Associative Law : A + B = B + A

b) Commutative Law : A + (B + C) = (A + B) + C

c) Existence of Identity : A + 0 = 0 + A = A

d) Existence of the Inverse : If A + X = 0, then, X = – A

Let $A = [a_{ij}]_{m \times n}$. If X is any Matrix of the same type such that A + X = 0, the zero matrix then, the Matrix X is called the "Additive Inverse" of the Matrix A.

(B) SUBTRACTION

Two Matrices A and B can be subtracted if and only if they have the same order or dimension i.e., the number of column of Matrix A is equal to the number of column of B and the number of Row of Matrix A is equal to the number of Row of B. In other words, Two Matrices of the same order are said to be Conformable for subtraction.

The subtraction (difference) of two matrices of the same order is obtained by subtracting corresponding elements. If $A = [a_{ij}]$ and $B = [b_{ij}]$, then C = A – B is the matrix having a general element of the form $c_{ij} = a_{ij} - b_{ij}$

Examples :

1. If $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 10 & 2 \\ 8 & 6 \end{bmatrix}$, find A – B and B – A.

Solution :

$$A - B = \begin{bmatrix} (1-10) & (5-2) \\ (6-8) & (7-6) \end{bmatrix} = \begin{bmatrix} -9 & 3 \\ -2 & 1 \end{bmatrix}_{2 \times 2}$$

$$B - A = \begin{bmatrix} (10-1) & (2-5) \\ (8-6) & (6-7) \end{bmatrix} = \begin{bmatrix} 9 & -3 \\ 2 & -1 \end{bmatrix}_{2 \times 2}$$

2. If A $= \begin{bmatrix} 3 & 7 \\ 4 & 8 \\ 2 & 1 \end{bmatrix}$ and B $= \begin{bmatrix} 2 & 5 \\ 6 & -3 \\ 4 & 11 \end{bmatrix}$, find A – B and B – A.

Solution :

$$A - B = \begin{bmatrix} (3-2) & (7-5) \\ (4-6) & 8-(-3) \\ (2-4) & (1-11) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 11 \\ -2 & -10 \end{bmatrix}_{3\times 2}$$

$$B - A = \begin{bmatrix} (2-3) & (5-7) \\ (6-4) & (-3-8) \\ (4-2) & (11-1) \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & -11 \\ 2 & 10 \end{bmatrix}_{3\times 2}$$

3. Let A $= \begin{bmatrix} 3 & 6 \\ 7 & 0 \end{bmatrix}$, B $= \begin{bmatrix} -1 & 7 \\ 8 & 4 \end{bmatrix}$ and C $= \begin{bmatrix} 5 & 4 \\ 1 & 9 \end{bmatrix}$, verify that (A+B) – C = A + (B–C).

Solution :

$$(A+B) = \begin{bmatrix} 3 & 6 \\ 7 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 7 \\ 8 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 13 \\ 15 & 4 \end{bmatrix}$$

$$(A+B) - C = \begin{bmatrix} 2 & 13 \\ 15 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 4 \\ 1 & 9 \end{bmatrix} = \begin{bmatrix} -3 & 9 \\ 14 & -5 \end{bmatrix}$$

$$(B - C) = \begin{bmatrix} -1 & 7 \\ 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 4 \\ 1 & 9 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 7 & -5 \end{bmatrix}$$

$$A + (B - C) = \begin{bmatrix} 3 & 6 \\ 7 & 0 \end{bmatrix} + \begin{bmatrix} -6 & 3 \\ 7 & -5 \end{bmatrix} = \begin{bmatrix} -3 & 9 \\ 14 & -5 \end{bmatrix}$$

$$(A + B) - C = \begin{bmatrix} -3 & 9 \\ 14 & -5 \end{bmatrix}$$

$$A + (B - C) = \begin{bmatrix} -3 & 9 \\ 14 & -5 \end{bmatrix}$$

Therefore, (A + B) – C = A + (B – C)

(C) MULTIPLICATION OR PRODUCT

(i) Multiplication of a Matrix by a Number

or

Scalar Multiplication

To multiply a Matrix A of order m × n by a scalar or a number 'K', we multiply every element in the Matrix A by the scalar K. The new Matrix thus obtained is the Matrix KA and its orders is also m × n.

Therefore, $KA = K[a_{ij}] = [Ka_{ij}]$.

Examples :

1. If A $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, find KA.

$$KA = K\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} Ka & Kb \\ Kc & Kd \end{bmatrix}$$

2. If A $= \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}$, find 2A.

$$2A = 2\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ 0 & 2 \end{bmatrix}$$

3. If A $= \begin{bmatrix} 2 & -3 & 4 \\ -6 & 8 & -3 \end{bmatrix}$, find – 5A.

$$-5A = \begin{bmatrix} -10 & 15 & -20 \\ 30 & -40 & 15 \end{bmatrix}$$

4. If A $= \begin{bmatrix} 2 & 3 \\ -1 & -4 \end{bmatrix}$, find –4A.

$$-4A = \begin{bmatrix} -8 & -12 \\ 4 & 16 \end{bmatrix}$$

ii) Multiplication or Product of a Row Matrix and a Column Matrix or Multiplication of Two Vectors

Two Matrices A and B (i.e., Column and Row) can be multiplied if and only if the number of columns of the first Matrix (i.e., A) must be equal to the number of rows of the second Matrix (i.e., B). In other words multiplication of two vectors A and B (i.e., Column and Row) is only possible when A and B are 'Conformable'. If the number of columns of the first Matrix is equal to the number of rows of the second Matrix then, the two Matrices A and B are said to be "Conformable" for multiplication.

In the Matrix Product AB, A is called the "Pre-Multiplier" (Prefactor) and B is called the "Post-Multiplier" (Post-factor).

Examples:

1. If $A = [a_1\ a_2\ a_3]_{1\times 3}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{3\times 1}$, find AB.

Solution :

AB is possible or defined, since the given two matrices A and B are Conformable. Therefore, in answer, order is 1 × 1

Therefore, $AB = [a_1b_1 + a_2b_2 + a_3b_3]_{1\times 1}$

The product is 1 × 1 Matrix.

2. If $A = [1\ 2\ 3]_{1\times 3}$ and $B = \begin{bmatrix} 8 \\ 5 \\ 3 \end{bmatrix}_{3\times 1}$, find AB.

Solution :

AB is defined, since the given two Matrices A and B are conformable.
Hence, in answer order is 1×1.

AB = [illegible]

The Product is 1×1 Matrix, i.e., single element.

3. If $A = \begin{bmatrix} 5 \\ 6 \\ 3 \\ 2 \end{bmatrix}_{4\times 1}$ and $B = [1\ 2\ 6\ 3\ 5]_{1\times 5}$, find AB.

Solution :

AB is possible, since the given two Matrices A and B are conformable.
Hence, in answer, order is 4×5.

$$AB = \begin{bmatrix} 5\times1 & 5\times2 & 5\times6 & 5\times3 & 5\times5 \\ 6\times1 & 6\times2 & 6\times6 & 6\times3 & 6\times5 \\ 3\times1 & 3\times2 & 3\times6 & 3\times3 & 3\times5 \\ 2\times1 & 2\times2 & 2\times6 & 2\times3 & 2\times5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 10 & 30 & 15 & 25 \\ 6 & 12 & 36 & 18 & 30 \\ 3 & 6 & 18 & 9 & 15 \\ 2 & 4 & 12 & 6 & 10 \end{bmatrix}_{4\times 5}$$

The Product is 4×5 Matrix.

4. If $A = [1\ 2\ 3\ 4]$ and $B = [5\ 6\ 7\ 8\ 9]$, find AB.

Solution:

AB is not possible or not defined, since the given two matrices are not conformable.

5. If $A = \begin{bmatrix} -4 \\ 9 \\ -7 \\ 3 \end{bmatrix}_{4\times 1}$ and $B = \begin{bmatrix} -1 \\ 5 \\ -7 \\ -7 \end{bmatrix}_{4\times 1}$, find AB.

Solution:

AB is not possible or defined, since the given two matrices are not conformable.

(iii) **Multiplication of a Matrix by a Matrix**

or

Product of Two Matrices or Matrix Multiplication

Two Matrices A and B can be multiplied if and only if the number of columns of the first Matrix (i.e.,A) must be equal to the number of rows of second Matrix (i.e.,B). In other words, Multiplication of two Matrices A and B is only possible when A and B are conformable. If $A = [a_{ij}]$ and $B = [b_{ij}]$ then AB [illegible]

Examples :

1. If A = [illegible] and B = [illegible], find AB.

Solution :

The given two Matrices A and B are conformable. Hence AB is possible or defined. Therefore, in answer, order is [illegible]. For multiplication of two Matrices, each element of the row is multiplied into the corresponding element of the column and then the products are summed.

AB = [illegible]

The product is [illegible] Matrix.

2. If $A = \begin{bmatrix} 5 & 6 & 2 \\ 7 & 8 & 2 \end{bmatrix}_{2\times 3}$ and $B = \begin{bmatrix} 2 & 0 \\ 1 & 5 \\ 3 & 1 \end{bmatrix}_{3\times 2}$, find AB.

Solution :

The given two matrices A and B are conformable. Therefore, AB is possible or defined. Hence, in answer, order is 2×2.

$$AB = \begin{bmatrix} 5\times2 + 6\times1 + 2\times3 & 5\times0 + 6\times5 + 2\times1 \\ 7\times2 + 8\times1 + 2\times3 & 7\times0 + 8\times5 + 2\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 10+6+6 & 0+30+2 \\ 14+8+6 & 0+40+2 \end{bmatrix} = \begin{bmatrix} 22 & 32 \\ 28 & 42 \end{bmatrix}_{2\times 2}$$

The product is 2×2 Matrix.

3. Find AB and BA, if $A = \begin{bmatrix} 4 & 6 & 2 \\ 1 & 7 & 4 \\ 3 & 9 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 8 \\ 7 \\ 1 \end{bmatrix}$

Solution :

AB can be defined, since the given two Matrices A and B are conformable. Hence, in answer order is 3×1.

$$AB = \begin{bmatrix} 4 & 6 & 2 \\ 1 & 7 & 4 \\ 3 & 9 & 2 \end{bmatrix}_{3\times 3} \begin{bmatrix} 8 \\ 7 \\ 1 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 32 + 42 + 2 \\ 8 + 49 + 4 \\ 24 + 63 + 2 \end{bmatrix} = \begin{bmatrix} 76 \\ 61 \\ 89 \end{bmatrix}_{3\times 1}$$

BA is not defined, since B and A are not conformable.

4. Verify whether AB = BA for the matrices

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution :

$$AB = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-2+0 & 4+0+0 & -2+1+0 \\ 1+2+2 & 2+0+2 & -1-1+4 \\ 0-2+3 & 0+0+3 & 0+1+6 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 4 & -1 \\ 5 & 4 & 2 \\ 1 & 3 & 7 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2+2+0 & 1-2-1 & 0+4-3 \\ -4+0+0 & -2+0+1 & 0+0+3 \\ 2+1+0 & 1-1+2 & 0+2+6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -2 & 1 \\ -4 & -1 & 3 \\ 3 & 2 & 8 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 4 & -1 \\ 5 & 4 & 2 \\ 1 & 3 & 7 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & -2 & 1 \\ -4 & -1 & 3 \\ 3 & 2 & 8 \end{bmatrix}$$

Therefore, AB = BA.

5. Find B, *if*, $-A + 2B = 6C$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & -1 & 7 \\ 3 & 0 & 0 \\ 4 & -1 & -5 \end{bmatrix}.$$

Solution :

$-A+2B = 6C$

$2B = 6C + A$

$B = \dfrac{6C+A}{2}$ or $B = \dfrac{6}{2}C + \dfrac{A}{2}$ or $B = 3C + \dfrac{1}{2}A$ or $B = \dfrac{1}{2}A + 3C.$

$$B = \frac{1}{2}\begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} + 3\begin{bmatrix} 2 & -1 & 7 \\ 3 & 0 & 0 \\ 4 & -1 & -5 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{3}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 6 & -3 & 21 \\ 9 & 0 & 0 \\ 12 & -3 & -15 \end{bmatrix} = \begin{bmatrix} 6\frac{1}{2} & -2 & 20\frac{1}{2} \\ 10\frac{1}{2} & 0 & \frac{1}{2} \\ 12\frac{1}{2} & -2\frac{1}{2} & -14\frac{1}{2} \end{bmatrix}$$

6. Given $A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \\ 7 & 8 & -9 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 \\ 7 & 0 \\ -9 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & -5 & 6 \end{bmatrix}$, verify A (BC) = (AB) C.

Solution :

$$BC = \begin{bmatrix} 4 & 3 \\ 7 & 0 \\ -9 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & -5 & 6 \end{bmatrix} = \begin{bmatrix} 4+6 & 8+0 & 12-15 & 16+18 \\ 7+0 & 14+0 & 21+0 & 28+0 \\ -9+4 & -18+0 & -27-10 & -36+12 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 8 & -3 & 34 \\ 7 & 14 & 21 & 28 \\ -5 & -18 & -37 & -24 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \\ 7 & 8 & -9 \end{bmatrix}\begin{bmatrix} 10 & 8 & -3 & 34 \\ 7 & 14 & 21 & 28 \\ -5 & -18 & -37 & -24 \end{bmatrix}$$

$$= \begin{bmatrix} 10+14+15 & 8+28+54 & -3+42+111 & 34+56+72 \\ 40-35-30 & 32-70-108 & -12-105-222 & 136-140-144 \\ 70+56+45 & 56+112+162 & -21+168+333 & 238+224+216 \end{bmatrix}$$

$$= \begin{bmatrix} 39 & 90 & 150 & 162 \\ -25 & -146 & -339 & -148 \\ 171 & 330 & 480 & 678 \end{bmatrix} \quad \text{—(1)}$$

$$AB = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \\ 7 & 8 & -9 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 7 & 0 \\ -9 & 2 \end{bmatrix} = \begin{bmatrix} 4+14+27 & 3+0-6 \\ 16-35-54 & 12+0+12 \\ 28+56+81 & 21+0-18 \end{bmatrix} = \begin{bmatrix} 45 & -3 \\ -73 & 24 \\ 165 & 3 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} 45 & -3 \\ -73 & 24 \\ 165 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & -5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 45-6 & 90+0 & 135+15 & 180-18 \\ -73+48 & -146+0 & -219-120 & -292+144 \\ 165+6 & 330+0 & 495-15 & 660+18 \end{bmatrix}$$

$$= \begin{bmatrix} 39 & 90 & 150 & 162 \\ -25 & -146 & -339 & -148 \\ 171 & 330 & 480 & 678 \end{bmatrix} \quad \text{—(2)}$$

Since (1) = (2), A (BC) = (AB) C.

7. Given $A = \begin{bmatrix} 8 & 1 & -2 \\ -9 & 9 & 9 \\ 6 & -3 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 3 \\ 5 & 6 & -4 \\ 7 & -9 & 8 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & -3 & 1 \\ 6 & 2 & -1 \\ 0 & -4 & 3 \end{bmatrix}$,

show that (i) A (B + C) = AB + AC

(ii) (A+B)C = AC + BC.

Solution :

i) $A(B+C) = \begin{bmatrix} 8 & 1 & -2 \\ -9 & 9 & 9 \\ 6 & -3 & 9 \end{bmatrix} \begin{bmatrix} 5 & -5 & 4 \\ 11 & 8 & -5 \\ 7 & -13 & 11 \end{bmatrix}$

$$= \begin{bmatrix} 40+11-14 & -40+8+26 & 32-5-22 \\ -45+99+63 & 45+72-117 & -36-45+99 \\ 30-33+63 & -30-24-117 & 24+15+99 \end{bmatrix}$$

$$= \begin{bmatrix} 37 & -6 & 5 \\ 117 & 0 & 18 \\ 60 & -171 & 138 \end{bmatrix} \quad \text{—(1)}$$

$$AB = \begin{bmatrix} 8 & 1 & -2 \\ -9 & 9 & 9 \\ 6 & -3 & 9 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 5 & 6 & -4 \\ 7 & -9 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 8+5-14 & -16+6+18 & 24-4-16 \\ -9+45+63 & 18+54-81 & -27-36+72 \\ 6-15+63 & -12-18-81 & 18+12+72 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 8 & 4 \\ 99 & -9 & 9 \\ 54 & -111 & 102 \end{bmatrix} \quad \text{—(2)}$$

$$AC = \begin{bmatrix} 8 & 1 & -2 \\ -9 & 9 & 9 \\ 6 & -3 & 9 \end{bmatrix} \begin{bmatrix} 4 & -3 & 1 \\ 6 & 2 & -1 \\ 0 & -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 32+6+0 & -24+2+8 & 8-1-6 \\ -36+54+0 & 27+18-36 & -9-9+27 \\ 24-18+0 & -18-6-36 & 6+3+27 \end{bmatrix}$$

$$= \begin{bmatrix} 38 & -14 & 1 \\ 18 & 9 & 9 \\ 6 & -60 & 36 \end{bmatrix} \quad \text{---- (3)}$$

$(2) + (3) = AB + AC = \begin{bmatrix} 37 & -6 & 5 \\ 117 & 0 & 18 \\ 60 & -171 & 138 \end{bmatrix}$ Therefore, A (B+C) = AB+AC.

(ii) (A+B) C = AC + BC

$$(A+B)\,C = \begin{bmatrix} 9 & -1 & 1 \\ -4 & 15 & 5 \\ 13 & -12 & 17 \end{bmatrix} \begin{bmatrix} 4 & -3 & 1 \\ 6 & 2 & -1 \\ 0 & -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 36-6+0 & -27-2-4 & 9+1+3 \\ -16+90+0 & 12+30-20 & -4-15+15 \\ 52-72+0 & -39-24-68 & 13+12+51 \end{bmatrix}$$

$$= \begin{bmatrix} 30 & -33 & 13 \\ 74 & 22 & -4 \\ -20 & -131 & 76 \end{bmatrix} \quad \text{—(1)}$$

$$AC = \begin{bmatrix} 8 & 1 & -2 \\ -9 & 9 & 9 \\ 6 & -3 & 9 \end{bmatrix} \begin{bmatrix} 4 & -3 & 1 \\ 6 & 2 & -1 \\ 0 & -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 32+6+0 & -24+2+8 & 8-1-6 \\ -36+54+0 & 27+18-36 & -9-9+27 \\ 24-18+0 & -18-6-36 & 6+3+27 \end{bmatrix} = \begin{bmatrix} 38 & -14 & 1 \\ 18 & 9 & 9 \\ 6 & -60 & 36 \end{bmatrix} \quad \text{—(2)}$$

$$BC = \begin{bmatrix} 1 & -2 & 3 \\ 5 & 6 & -4 \\ 7 & -9 & 8 \end{bmatrix} \begin{bmatrix} 4 & -3 & 1 \\ 6 & 2 & -1 \\ 0 & -4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 4-12+0 & -3-4-12 & 1+2+9 \\ 20+36+0 & -15+12+16 & 5-6-12 \\ 28-54+0 & -21-18-32 & 7+9+24 \end{bmatrix} = \begin{bmatrix} -8 & -19 & 12 \\ 56 & 13 & -13 \\ -26 & -71 & 40 \end{bmatrix} \quad \text{—(3)}$$

$(2) + (3) = AC + BC = \begin{bmatrix} 30 & -33 & 13 \\ 74 & 22 & -4 \\ -20 & -131 & 76 \end{bmatrix}$

Therefore, (A+B) C = AC+BC.

8. Let $A = \begin{bmatrix} -1 & 3 & 1 \\ 0 & -2 & 4 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Calculate AI and IX.

Solution:

$$AI = \begin{bmatrix} -1 & 3 & 1 \\ 0 & -2 & 4 \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

$$= \begin{bmatrix} -1+0+0 & 0+3+0 & 0+0+1 \\ 0+0+0 & 0-2+0 & 0+0+4 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 1 \\ 0 & -2 & 4 \end{bmatrix}_{2\times 3}$$

$$IX = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1} = \begin{bmatrix} x_1 + 0 \\ 0 + x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Answer:

$$AI = \begin{bmatrix} -1 & 3 & 1 \\ 0 & 2 & 4 \end{bmatrix} \qquad Ix = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

9. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, Show that $A^2 - 5A + 7I = 0$.

Solution:

$$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$5A = 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$$

$$7I = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$A^2 - 5A = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} = \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix}$$

$$A^2 - 5A + 7I = \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

PROPERTIES OF MATRIX MULTIPLICATION

a) Matrix Multiplication is associative.
   i.e., $A(BC) = (AB)C$.

b) Matrix Multiplication is not always Commutative.
   i.e., $AB \neq BA$.

c) Multiplication of Matrices is distributive with respect to Addition of Matrices.
   i.e., $A(B+C) = AB+AC$.

d) $AO = OA = O$.

TRACE

The sum of diagonal elements of a Square Matrix A $(a_{11} + a_{22} + \ldots + a_{nn})$ is called the "Trace of A". The Trace of 'A' is denoted by Tr (A).

Example:

If $A = \begin{bmatrix} 3 & 5 & 7 \\ 7 & 9 & 4 \\ 9 & -7 & -5 \end{bmatrix}$, then $Tr(A) = 3 + 9 - 5 = 7$.

We observe that the Trace is defined only for a "Square Matrix".

The Trace satisfies the following Properties.

1) $Tr(A+B) = Tr(A) + Tr(B)$
2) $Tr(aA) = a\,Tr(A)$
3) $Tr(AB) = Tr(BA)$
4) $Tr(A) = Tr(CAC^{-1})$, when C is a "Non–Singular Matrix".

TRANSPOSE OF A MATRIX

The Transpose of a Matrix 'A' is a New Matrix in which the rows and columns of Matrix A have been interchanged. That is, rows should be transferred or converted into columns and columns should be transferred or converted into rows. Transpose of a Matrix is denoted by A' or A^t. Therefore, if $A = (a_{ij})$
A' or $A^t = (a_{ji})$

Examples:

1. If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, find A^t.

$$A^t = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

2. Find A^T, if $A = \begin{bmatrix} 3 & 7 \\ 2 & 9 \\ 5 & 11 \end{bmatrix}$.

Therefore, $A^T = \begin{bmatrix} 3 & 2 & 5 \\ 7 & 9 & 11 \end{bmatrix}$.

3. If $A = \begin{bmatrix} 3 & 8 & -9 \\ 5 & 7 & 9 \end{bmatrix}$, find A^T.

$$A^T = \begin{bmatrix} 3 & 5 \\ 8 & 7 \\ -9 & 9 \end{bmatrix}.$$

Properties

A) The Transpose of the Transpose of a Matrix is the Original Matrix. i.e., $(A^T)^T = A$.

Examples:

1. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$, $(A^T)^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

2. If $A = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$, $A^T = [3\ 7\ 5]$, $(A^T)^T = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$.

B) The Transpose of a sum of two Matrices (A and B) is the sum of the transposes of the individual Matrices.

i.e., $(A+B)^T = A^T + B^T$.

Examples:

1. If $A = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ and $B = \begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}$, verify $(A+B)^T = A^T + B^T$.

$$A + B = \begin{bmatrix}6 & 8\\10 & 12\end{bmatrix}, \quad (A+B)^T = \begin{bmatrix}6 & 10\\8 & 12\end{bmatrix}$$

$$A^T + B^T = \begin{bmatrix}1 & 3\\2 & 4\end{bmatrix} + \begin{bmatrix}5 & 7\\6 & 8\end{bmatrix} = \begin{bmatrix}6 & 10\\8 & 12\end{bmatrix}$$

Therefore, $(A+B)^T = A^T + B^T$.

2. If $A = \begin{bmatrix}3 & 5 & -7\\9 & -3 & 3\\7 & -9 & 1\end{bmatrix}$ and $B = \begin{bmatrix}-1 & 3 & 5\\7 & 9 & 1\\-3 & 5 & -5\end{bmatrix}$, verify that $(A+B)^T = A^T + B^T$.

$$A+B = \begin{bmatrix}2 & 8 & -2\\16 & 6 & 4\\4 & -4 & -4\end{bmatrix}, \quad (A+B)^T = \begin{bmatrix}2 & 16 & 4\\8 & 6 & -4\\-2 & 4 & -4\end{bmatrix}$$

$$A^T+B^T = \begin{bmatrix}3 & 9 & 7\\5 & -3 & -9\\-7 & 3 & 1\end{bmatrix} + \begin{bmatrix}-1 & 7 & -3\\3 & 9 & 5\\5 & 1 & -5\end{bmatrix} = \begin{bmatrix}2 & 16 & 4\\8 & 6 & -4\\-2 & 4 & -4\end{bmatrix}$$

Therefore, $(A+B)^T = A^T + B^T$.

C) The Transpose of a product of Matrices is the product in reverse order of their transposes.

i.e., $(AB)^T = B^T A^T$ and $(ABC)^T = C^T B^T A^T$.

Examples:

1) If $A = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ and $B = \begin{bmatrix}0 & -1\\6 & 7\end{bmatrix}$, verify that $(AB)^T = B^T A^T$.

$$AB = \begin{bmatrix}1\times0 + 2\times6 & 1\times-1 + 2\times7\\3\times0 + 4\times6 & 3\times-1 + 4\times7\end{bmatrix} = \begin{bmatrix}0 + 12 & -1 + 14\\0 + 24 & -3 + 28\end{bmatrix}$$

$$= \begin{bmatrix}12 & 13\\24 & 25\end{bmatrix} \quad (AB)^T = \begin{bmatrix}12 & 24\\13 & 25\end{bmatrix}.$$

$$B^T = \begin{bmatrix}0 & 6\\-1 & 7\end{bmatrix} \quad A^T = \begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}$$

$$B^TA^T = \begin{bmatrix}0\times1 + 6\times2 & 0\times3 + 6\times4\\-1\times1 + 7\times2 & -1\times3 + 7\times4\end{bmatrix} = \begin{bmatrix}0+12 & 0+24\\-1+14 & -3+28\end{bmatrix} = \begin{bmatrix}12 & 24\\13 & 25\end{bmatrix}$$

Therefore, $(AB)^T = B^T A^T$.

2. If $A = \begin{bmatrix}2 & 1\\4 & 3\\1 & 0\end{bmatrix}$ and $B = \begin{bmatrix}1 & 3\\2 & 2\end{bmatrix}$, verify that $(AB)^T = B^T A^T$.

$$AB = \begin{bmatrix}2\times1 + 1\times2 & 2\times3 + 1\times2\\4\times1 + 3\times2 & 4\times3 + 3\times2\\1\times1 + 0\times2 & 1\times3 + 0\times2\end{bmatrix} = \begin{bmatrix}2 + 2 & 6 + 2\\4 + 6 & 12 + 6\\1 + 0 & 3 + 0\end{bmatrix}$$

$$= \begin{bmatrix}4 & 8\\10 & 18\\1 & 3\end{bmatrix}$$

$$(AB)^T = \begin{bmatrix}4 & 10 & 1\\8 & 18 & 3\end{bmatrix}$$

$$B^T = \begin{bmatrix}1 & 2\\3 & 2\end{bmatrix} \quad A^T = \begin{bmatrix}2 & 4 & 1\\1 & 3 & 0\end{bmatrix}$$

$$B^TA^T = \begin{bmatrix}1\times2 + 2\times1 & 1\times4 + 2\times3 & 1\times1 + 2\times0\\3\times2 + 2\times1 & 3\times4 + 2\times3 & 3\times1 + 2\times0\end{bmatrix}$$

$$= \begin{bmatrix}2 + 2 & 4 + 6 & 1 + 0\\6 + 2 & 12 + 6 & 3 + 0\end{bmatrix} = \begin{bmatrix}4 & 10 & 1\\8 & 18 & 3\end{bmatrix}$$

Therefore, $(AB)^T = B^T A^T$.

D) Transpose of a Product of Scalar (i.e., a Complex Number) and a Matrix is the product of the scalar and transpose of the Matrix.

i.e., $(KA)^T = KA^T$.

Examples:

1. If $A = \begin{bmatrix}7 & 8\\9 & 10\end{bmatrix}$ and K is 5, verify $(KA)^T = KA^T$.

$$KA = 5\begin{bmatrix}7 & 8\\9 & 10\end{bmatrix} = \begin{bmatrix}35 & 40\\45 & 50\end{bmatrix} \quad (KA)^T = \begin{bmatrix}35 & 45\\40 & 50\end{bmatrix}$$

$$A^T = \begin{bmatrix}7 & 9\\8 & 10\end{bmatrix} \quad KA^T = 5\begin{bmatrix}7 & 9\\8 & 10\end{bmatrix} = \begin{bmatrix}35 & 45\\40 & 50\end{bmatrix}$$

Therefore, $(KA)^T = KA^T$.

2. If $A = \begin{bmatrix}2 & 2\\1 & 4\end{bmatrix}$ and K is 9, verify that $(KA)^T = KA^T$.

$$A = \begin{bmatrix}2 & 2\\1 & 4\end{bmatrix} \quad \text{Therefore, } KA = 9\begin{bmatrix}2 & 2\\1 & 4\end{bmatrix} = \begin{bmatrix}18 & 18\\9 & 36\end{bmatrix}$$

$$(KA)^T = \begin{bmatrix}18 & 9\\18 & 36\end{bmatrix}$$

$$A^T = \begin{bmatrix}2 & 1\\2 & 4\end{bmatrix} \quad \text{Therefore, } KA^T = 9\begin{bmatrix}2 & 1\\2 & 4\end{bmatrix} = \begin{bmatrix}18 & 9\\18 & 36\end{bmatrix}$$

Therefore, $(KA)^T = KA^T$.

The Conjugate of a Matrix

The Martix obtained from any given Matrix A, on replacing its elements by the corresponding conjugate complex numbers is called the "Conjugate Matrix" or " Conjugate of the Matrix A" and is denoted by $\bar{A}$.

Thus, if $A = [a_{ij}]_{m\times n}$, then $\bar{A} = [\bar{a}_{ij}]_{m\times n}$,
where,

$\bar{a}_{ij}$ is the conjugate complex of a_{ij}.

If, $\bar{A}$ be a matrix over the field of real numbers, then

$\bar{A} = A$.

The Conjugate Transpose of a Matrix

The Conjugate of the Transpose of Matrix A is called the " Conjugate Transpose of A" and is denoted by A.

Obviously, the conjugate of the transpose is the same as the transpose of the conjugate.

i.e., $(\bar{A}^T) = (\bar{A})^T = A$

Example :

1. If $A = \begin{bmatrix} 1 & 2+i & 3+2i \\ 3-i & 3 & -3i \\ 3-2i & 3i & -2 \end{bmatrix}$ then,

$\bar{A} = \begin{bmatrix} 1 & 2-i & 3-2i \\ 3+i & 3 & 3i \\ 3+2i & -3i & -2 \end{bmatrix}$ $(\bar{A})^T = \begin{bmatrix} 1 & 2+i & 3+2i \\ 3-i & 3 & -3i \\ 3-2i & 3i & -2 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 3-i & 3-2i \\ 2+i & 3 & 3i \\ 3+2i & -3i & -2 \end{bmatrix}$ and $(\bar{A}^T) = \begin{bmatrix} 1 & 2+i & 3+2i \\ 3-i & 3 & -3i \\ 3-2i & 3i & -2 \end{bmatrix}$.

DETERMINANTS

A Determinant is a number associated with a Square Matrix. It is denoted by |A|.

A) First Order Matrix

For example, if A = [3], then |A| = 3.

B) Second Order Matrix

The value of a Determinant of a Second Order Matrix is equal to the product of the principal diagonal elements minus the product of off–diagonal elements.

Examples:

1. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, find |A|. $|A| = a_{11}a_{22} - a_{21}a_{12}$.

2. If $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$, find |A|. |A| = 4+6 = 10.

3. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, find |A|. |A| = 4–4 = 0.

C) Third Order Matrix

Examples:

1. If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, find |A|.

$$|A| = a_{11}\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}.$$

2. If $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, find |A|.

$$|A| = 2\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 1\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} + 3\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$= 2(5\times 9 - 6\times 8) - 1(4\times 9 - 7\times 6) + 3(4\times 8 - 7\times 5)$

$|A| = 2(45-48) - 1(36-42) + 3(32-35)$

$= 2(-3) - 1(-6) + 3(-3) = -6 + 6 - 9 = -9.$

3. Evaluate $A = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{vmatrix} = -12.$

$|A| = 1(3-2) - 2(2-3) + 3(4-9) = 1 + 2 - 15 = -12.$

4. If $A = \begin{bmatrix} 4 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 4 \end{bmatrix}$, find |A|.

$|A| = 4(4-0) - 2(0) + 4(0) = 4(4) - 0 + 0 = 16$

[i.e., product of the leading diagonal elements, since given Matrix is a Triangular Matrix].

5. Find |A|, if $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 2 & 1 \end{bmatrix}$.

$|A| = 2(2-0) - 0 + 0 = 2\times 2 = 4$ [i.e., the product of the leading diagonal elements, since the given Matrix is a Triangular Matrix].

D) Fourth Order Matrix

1) If $A = \begin{bmatrix} 2 & 2 & 1 & 4 \\ 8 & 7 & 1 & 6 \\ 1 & 8 & 6 & 6 \\ 9 & 9 & 0 & 9 \end{bmatrix}$ find |A|.

$$|A| = 2\begin{bmatrix} 7 & 1 & 6 \\ 8 & 6 & 6 \\ 9 & 0 & 9 \end{bmatrix} - 2\begin{bmatrix} 8 & 1 & 6 \\ 1 & 6 & 6 \\ 9 & 0 & 9 \end{bmatrix} + 1\begin{bmatrix} 8 & 7 & 6 \\ 1 & 8 & 6 \\ 9 & 9 & 9 \end{bmatrix} - 4\begin{bmatrix} 8 & 7 & 1 \\ 1 & 8 & 6 \\ 9 & 9 & 0 \end{bmatrix}$$

a) $= 2\{(7(54-0) - 1(72-54) + 6(0-54)\}$
$= 2\{7(54) - 1(18) + 6(-54)\} = 2(378 - 18 - 324) = 2(36) = 72.$

b) $= -2\{8(54-0) - 1(9-54) + 6(0-54)\}$
$= -2\{8(54) - 1(-45) + 6(-54)\}$
$= -2\{432 + 45 - 324\} = -2(153) = -306.$

c) $= 1\{8(72-54) - 7(9-54) + 6(9-72)\}$
$= 1\{8(18) - 7(-45) + 6(-63)\}$
$= 1(144 + 315 - 378) = 1(459) - 378 = 81.$

d) $= -4\{8(0-54) - 7(0-54) + 1(9-72)\}$
$= -4\{8(-54) - 7(-54) + 1(-63)\}$
$= -4(-432 + 378 - 63) = -4(-117) = 468.$

Therefore, $|A| = 72 - 306 + 81 + 468 = 315.$

Properties of Determinant

A) If any two adjacent rows (or columns) of a determinant are interchanged, the value of the determinant changes only in sign.

Examples :

1. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, $|A| = 2 - 6 = -4$

   Matrix after interchanging rows

   i.e., $B = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ $|B| = 6 - 2 = 4.$

2. If $A = \begin{bmatrix} 4 & 8 \\ 6 & 5 \end{bmatrix}$, $|A| = 20 - 48 = -28$

   Matrix after interchanging columns

   i.e., $B = \begin{bmatrix} 8 & 4 \\ 5 & 6 \end{bmatrix}$, $|B| = 48 - 20 = 28.$

B) If the rows and columns of a determinant are interchanged (Transposed Matrix), the value of the determinant does not change.

or

The determinant of the Matrix is the same as the determinant of the Transposed Matrix. i.e., $|A^T| = |A|$

Examples :

1. If $A = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}$, $|A| = 70 - 72 = -2.$

   $A^T = \begin{bmatrix} 7 & 9 \\ 8 & 10 \end{bmatrix}$ $|A^T| = 70 - 72 = -2.$

2. If $A = \begin{bmatrix} 4 & 6 \\ 3 & 1 \end{bmatrix}$, $|A| = 4 - 18 = -14.$

   $A^T = \begin{bmatrix} 4 & 3 \\ 6 & 1 \end{bmatrix}$, $|A^T| = 4 - 18 = -14.$

C) If all the elements of any row or column of a determinant are multiplied by a scalar (K), the value of the new determinant is K times that of the original.

or

If all the elements of one row or of one column are multiplied by the same quantity, then the determinant is multiplied by the same quantity.

Examples :

1. If $A = \begin{bmatrix} 7 & 4 \\ 2 & 1 \end{bmatrix}$, $|A| = 7 - 8 = -1.$

If we multiply the first row by 3 (i.e., $R_1 \times 3$), the new Matrix is $\begin{bmatrix} 21 & 12 \\ 2 & 1 \end{bmatrix}$

Determinant of the new Matrix = 21–24 = – 3.

2. If $A = \begin{bmatrix} 8 & 3 \\ 6 & 4 \end{bmatrix}$, $|A| = 32 - 18 = 14.$

If we multiply the second row by 4 (i.e., $R_2 \times 4$) the new Matrix is $\begin{bmatrix} 8 & 3 \\ 24 & 16 \end{bmatrix}$

Determinant of the new Matrix = 128 – 72 = 56.

3. If $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$, $|A| = 15 - 14 = 1.$

If we multiply the first column by 4 (i.e., $C_1 \times 4$) the new M atrix is $\begin{bmatrix} 20 & 2 \\ 28 & 3 \end{bmatrix}$

Determinant of the new Matrix = 60 – 56 = 4.

4. If $A = \begin{bmatrix} 2 & 9 \\ 1 & 3 \end{bmatrix}$, $|A| = 6 - 9 = -3.$

If we multiply the second column by 7 (i.e., $C_2 \times 7$), the new Matrix is $\begin{bmatrix} 2 & 63 \\ 1 & 21 \end{bmatrix}$

Determinant of the new Matrix = 42 − 63 = − 21.

D) If any Matrix having the same row and same column (identical), the value of the determinant is zero.

Examples:

1. If $A = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$, $|A| = 8 - 8 = 0$.
2. If $A = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$, $|A| = 6 - 6 = 0$.
3. If $A = \begin{bmatrix} 3 & 7 & 3 \\ 6 & 2 & 6 \\ 3 & 7 & 3 \end{bmatrix}$, $|A| = 3(6 - 42) - 7(18 - 18) + 3(42 - 6)$

$= 3(-36) + 3(36) = -108 + 108 = 0.$

E) If one row (or column) of a Matrix is multiple of another row (or column), the value of the determinant will be zero.

Examples:

1. If $A = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$, $|A| = 18 - 18 = 0$.
2. If $A = \begin{bmatrix} 4 & 16 \\ 5 & 20 \end{bmatrix}$, $|A| = 80 - 80 = 0$.

F) If the addition or subtraction of any row (or column) to another row (or column), the value of determinant remains unchanged.

Examples :

1. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, $|A| = 0 - 3 = -3$.

The new Matrix after the addition of row $(R_1 + R_2)$ $= \begin{bmatrix} 2 & 3 \\ 1+2 & 0+3 \end{bmatrix}$

$= \begin{bmatrix} 2 & 3 \\ 3 & 3 \end{bmatrix}$.

Determinant of the new Matrix = 6−9 = − 3.

2. If $A = \begin{bmatrix} 7 & 6 \\ 5 & 4 \end{bmatrix}$, $|A| = 28 - 30 = -2$.

The new Matrix after the addition of columns $(C_1 + C_2) = \begin{bmatrix} 7 & 6+7 \\ 5 & 4+5 \end{bmatrix} = \begin{bmatrix} 7 & 13 \\ 5 & 9 \end{bmatrix}$

Determinant of new Matrix = 63−65 = − 2.

3. If $A = \begin{bmatrix} 5 & 8 \\ 6 & 7 \end{bmatrix}$, $|A| = 35 - 48 = -13$.

The new Matrix after the subtraction of rows $(R_2 - R_1)$ $= \begin{bmatrix} 5 & 8 \\ 6-5 & 7-8 \end{bmatrix}$

$= \begin{bmatrix} 5 & 8 \\ 1 & -1 \end{bmatrix}$.

Determinant of the new Matrix = −5 −8 = − 13.

4. If $A = \begin{bmatrix} 8 & 3 \\ 2 & 4 \end{bmatrix}$, $|A| = 32 - 6 = 26$.

The new Matrix after the subtraction of columns $(C_2 - C_1)$

$= \begin{bmatrix} 8 & 3-8 \\ 2 & 4-2 \end{bmatrix} = \begin{bmatrix} 8 & -5 \\ 2 & 2 \end{bmatrix}$

Determinant of the new Matrix = 16−(−10) = 16 + 10 = 26.

G) If any row (or column) consists entirely of zeros, then the determinant will be zero.

Examples :

1. If $A = \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix}$, $|A| = 0 - 0 = 0$. 2) If $A = \begin{bmatrix} 0 & 7 \\ 0 & 3 \end{bmatrix}$, $|A| = 0 - 0 = 0$.

H) If a Matrix is a Triangular Matrix, then the determinant is equal to the product of the elements on the leading diagonal.

Examples :

1. If $A = \begin{bmatrix} 4 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 4 \end{bmatrix}$, find |A|.

$|A| = 4(4-0) - 2(0) + 4(0) = 4(4) - 0 + 0 = 16$ (i.e., the product of the leading diagonal elements $4 \times 1 \times 4$).

2. If $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, find |A|.

$|A| = 2(2-0) - 0 + 0 = 2(2) = 4$ (i.e., the product of the leading diagonal elements $2 \times 2 \times 1$)

I) Determinant of the product of Matrices is equal to the product of the individual determinants.

i.e., $|AB| = |A|\,|B|$.

Examples :

1. If $A = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$, verify $|AB| = |A|\,|B|$.

$$AB = \begin{bmatrix} 2+14 & 6+18 \\ 1+14 & 3+18 \end{bmatrix} = \begin{bmatrix} 16 & 24 \\ 15 & 21 \end{bmatrix}, \ |AB| = 336-360 = -24$$

$|A| = 4-2 = 2$ $|B| = 9-21 = -12$ Therefore, $|A|\,|B| = 2 \times (-12) = -24$

Therefore, $|AB| = |A|\,|B|$.

2. If $A = \begin{bmatrix} 1 & 8 \\ 6 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 2 \\ 0 & 9 \end{bmatrix}$, verify $|AB| = |A|\,|B|$.

$$AB = \begin{bmatrix} 1\times7 + 8\times0 & 1\times2 + 8\times9 \\ 6\times7 + 6\times0 & 6\times2 + 6\times9 \end{bmatrix} = \begin{bmatrix} 7+0 & 2+72 \\ 42+0 & 12+54 \end{bmatrix} = \begin{bmatrix} 7 & 74 \\ 42 & 66 \end{bmatrix}$$

$|AB| = 462-3108 = -2646$

$|A| = 6-48 = -42$ and $|B| = 63-0 = 63$

Therefore, $|A|\,|B| = -42 \times 63 = -2646$. Therefore, $|AB| = |A|\,|B|$.

RANK OF A MATRIX

The Rank of a Matrix A is defined as the maximum number of linearly independent rows (or columns) in Matrix A. That is the number of Non-zero Vectors (Non-singular Matrix). Rank of a Matrix is denoted by r (A). The Rank of a Matrix cannot exceed the number of rows or columns whichever is smaller. The rank of "Null Matrix" is zero.

Rank of First Order Matrix

Examples:

1. If A = [3], then r (A) = 1.
2. If A = [0], then r (A) = 0.
3. If A = [9], then r (A) = 1.

Rank of Second Order Matrix

Examples:

1. Find the rank of Matrix

$$A = \begin{bmatrix} 5 & 2 \\ -2 & -3 \end{bmatrix}.$$

$|A| = -15-(-4) = -15+4 = -11$.

The determinant of the given Matrix is −11. Hence, this given Matrix contains a non singular Matrix of order 2.

Therefore, the rank of Matrix A i.e., r (A) = 2.

2. Find the rank of Matrix $A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$ $|A| = 12-12 = 0$

Since the given Matrix is singular, let us consider a first order Matrix (submatrix). Then the determinant of the first order Matrix is non-zero. Therefore, r (A) is 1.

3. Find the rank of a Matrix $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$|A| = 0-0 = 0$.

Since the determinant is zero, i.e., the given matrix is singular, r (A) is 0.

4. Find the rank of Matrix $A = \begin{bmatrix} -4 & 0 & 5 \\ 1 & 2 & 3 \end{bmatrix}$.

Since the given Matrix is not a square Matrix, Determinant cannot be defined. Therefore, let us consider the submatrices of order 2.

$\begin{bmatrix} -4 & 0 \\ 1 & 2 \end{bmatrix}$ Determinant $= -8-0 = -8 \neq 0$.

Therefore, r (A) = 2.

Rank of Third Order Matrix

Examples:

1. Find the rank of Matrix $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$.

$|A| = 1(0-0) - 4(0-0) + 0(12-15) = 0-0+0 = 0$.

Since the determinant is zero i.e., the given Matrix is singular, let us consider the submatrices of order 2.

$\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$ Determinant $= 5-8 = -3 \neq 0$.

Therefore, rank r (A) = 2.

2. Find the rank of a Matrix $A = \begin{bmatrix} 2 & 3 & 5 & 1 \\ 1 & 2 & 3 & 2 \\ 1 & 3 & 4 & 5 \end{bmatrix}$.

Since the given Matrix is not a square Matrix, determinant cannot be defined. Therefore, let us consider 3x3 Matrix.

i) $\begin{vmatrix} 2 & 3 & 5 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{vmatrix} = 2(8-9) - 3(4-3) + 5(3-2) = -2-3+5 = 0$

ii) $\begin{vmatrix} 3 & 5 & 1 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{vmatrix} = 3(15-8) - 5(10-6) + 1(8-9) = 21-20-1 = 0$

iii) $\begin{vmatrix} 2 & 5 & 1 \\ 1 & 3 & 2 \\ 1 & 4 & 5 \end{vmatrix} = 2(15-8) - 5(5-2) + 1(4-3) = 14-15+1 = 0$.

(v) $\begin{vmatrix} 3 & 3 & 1 \\ 3 & 2 & 2 \\ 3 & 3 & 4 \end{vmatrix}$ = [illegible]

Since determinants of the submatrices of order 3 are equal to zero, let us consider 2 × 2 Matrix

$\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 4 - 9 = 5$ [illegible]

Therefore the rank of Matrix r(A) = 2.

Rank of Fourth Order Matrix

Examples:

1. Find the rank of a Matrix $A = \begin{bmatrix} 8 & 0 & 2 & 2 \\ 3 & 1 & 0 & 2 \\ 4 & 1 & 2 & 2 \\ 3 & 2 & 3 & 2 \end{bmatrix}$

First, find the determinant of 4 × 4 Matrix.

i) $|A| = (8)[1(4-4) - 0(2-4) + 2(2-4)] = (8) \times (-4) = -24$

ii) $|A| = 0$

iii) $|A| = 2\begin{bmatrix} 3 & 1 & 2 \\ 4 & 1 & 2 \\ 3 & 2 & 2 \end{bmatrix}$

$= 2[3(2-4) - 1(8-6) + 2(8-3)] = (2)(-6-4+12)$
$= 2 \times 2 = 4.$

iv) $|A| = 2\begin{bmatrix} 3 & 1 & 0 \\ 4 & 1 & 2 \\ 3 & 2 & 2 \end{bmatrix}$

$= 2[3(2-4) - 1(8-6) + 0(8-3)] = (2)(-6-4)$
$= 2 \times (-10) = -20.$

Therefore Determinant of 4 × 4 Matrix = −24 − 0 + 4 − 20 = −40.

Since, the determinant is −40 the rank of a Matrix i.e., r(A) is 4

2. Find the rank of a Matrix $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

First, find the determinant for 4 × 4 Matrix.

Therefore, $|A| = 1[1(1-0) - 0(0-0) + 0(0-0)]$
$= 1 \times 1 = 1$

Since the determinant is 1, the rank of Matrix i.e., r(A) is 4.

MINOR

Let $A = (a_{ij})$ be an n × n Matrix (i.e., a Square Matrix).

If we delete the row and the column containing the element a_{ij}, we obtain a Square Matrix of order n − 1 and the Determinant of this square Matrix is called the "Minor" of the element a_{ij} and is denoted by M_{ij}. In other words the determinant obtained by omitting the row and column containing a particular element is called as the "Minor" of that element.

Example:

$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

In the above example, the Determinant

$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ is called the Minor of a_{11} i.e., M_{11}

$\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ is called the Minor of a_{12} i.e., M_{12}

$\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$ is called the Minor of a_{13} i.e., M_{13}

$\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$ is called the Minor of a_{21} i.e., M_{21}

$\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$ is called the Minor of a_{22} i.e., M_{22}

$\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$ is called the Minor of a_{23} i.e., M_{23}

$\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$ is called the Minor of a_{31} i.e., M_{31}

$\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$ is called the Minor of a_{32} i.e., M_{32}

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is called the Minor of a_{33} i.e., M_{33}

Examples :

1. Compute Minor for every element of the Matrix $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

Solution:

$M_{11} = 4, \ M_{12} = 3, \ M_{21} = 1, \ M_{22} = 2.$

2) Compute Minor for every element of the Matrix A = [illegible]

Solution:

M_{11} = [illegible] = 6, M_{12} = [illegible] = 3, M_{13} = [illegible] = 0

M_{21} = [illegible] = [illegible], M_{22} = [illegible] = [illegible], M_{23} = [illegible] = [illegible]

M_{31} = [illegible] = [illegible], M_{32} = [illegible] = [illegible] and M_{33} = [illegible] = [illegible]

COFACTOR

The Minor M_{ij} multiplied by [illegible] is called the Cofactor of the element a_{ij}. Cofactor is denoted by A_{ij}. In other words a Cofactor Matrix in which every element of a Matrix a_{ij} is replaced by its cofactor A_{ij}.

Therefore $A_{ij} = (-1)^{i+j} M_{ij}$. Hence, the signed Minors are called Cofactors.

Example:

A = [illegible]

In the above Matrix A

The Cofactor of the element a_{11} i.e., A_{11} = [illegible]

The Cofactor of the element a_{12} i.e., A_{12} = [illegible]

The Cofactor of the element a_{13} i.e., A_{13} = [illegible]

The Cofactor of the element a_{21} i.e., A_{21} = [illegible]

The Cofactor of the element a_{22} i.e., A_{22} = [illegible]

The Cofactor of the element a_{23} i.e., A_{23} = [illegible]

The Cofactor of the element a_{31} i.e., A_{31} = [illegible]

The Cofactor of the element a_{32} i.e., A_{32} = [illegible]

The Cofactor of the element a_{33} i.e., A_{33} = [illegible]

Examples:

1) Compute Cofactor for the Matrix A = [illegible]

Minors

M_{11} = [illegible] ⟶ A_{11} = + [illegible] = [illegible]

M_{12} = [illegible] ⟶ A_{12} = − [illegible] = [illegible]

M_{13} = [illegible] ⟶ A_{13} = + [illegible] = [illegible]

M_{21} = [illegible] ⟶ A_{21} = − [illegible] = [illegible]

M_{22} = [illegible] ⟶ A_{22} = + [illegible] = [illegible]

M_{23} = [illegible] ⟶ A_{23} = − [illegible] = [illegible]

M_{31} = [illegible] ⟶ A_{31} = + [illegible] = [illegible]

M_{32} = [illegible] ⟶ A_{32} = − [illegible] = [illegible]

M_{33} = [illegible] ⟶ A_{33} = + [illegible] = [illegible]

Therefore, Cofactor of A = [illegible]

2) Compute the Cofactor for the Matrix

A = [illegible]

Minor i.e., M_{ij} = [illegible] Cofactor i.e., A_{ij} = [illegible]

3) Compute the Cofactor for the Matrix A = [illegible]

Minor i.e., M_{ij} = [illegible] Cofactor i.e., A_{ij} = [illegible]

4) Compute the Cofactor for the Matrix A = [illegible]

Cofactor i.e., A_{ij} = [illegible]

ADJOINT OF THE MATRIX OR ADJUGATE OF THE MATRIX

The Transpose of a Cofactor Matrix (A_{ij}) is called the "Adjoint of the Matrix A" or "Adjugate of the Matrix A" and is denoted by **"Adj A"**.

That is to get Adjoint of a Matrix, rows of the Cofactor Matrix should be converted into columns and columns of the Cofactor Matrix should be converted into rows.

INVERSE OF A SQUARE MATRIX OR MATRIX INVERSION

A Matrix 'B' is said to be the "Inverse of a Square Matrix A," if it satisfies the following property :

AB = BA = I (Identity Matrix) i.e., $AA^{-1} = A^{-1}A = I$. In other words, one matrix is the inverse of the another if and only if (iff) their product is the Identity Matrix. Inverse of a Square Matrix is denoted by A^{-1}. Inverse of a Matrix is also called "Reciprocal Matrix". Only Square Matrices possess inverses. The necessary and sufficient condition for a Square Matrix 'A' to possess an inverse is that,

$|A| \neq 0$ i.e., A is a Non–Singular Matrix.

Computation or Formula for the determination of A^{-1}

$$A^{-1} = \frac{1}{|A|} \text{Adj } A \text{ or } A^{-1} = \frac{\text{Adj } A}{|A|}$$

Steps to Compute Inverse Matrix

1) Check that the given Matrix is a Square.
2) Find |A| of the given Matrix. If $|A| \neq 0$ go to the step 3. If |A| = 0 , we cannot find inverse.
3) Compute Minors (M_{ij})
4) Compute Cofactor Matrix (A_{ij})
5) Compute Adjoint Matrix (Adj A) i.e., Transpose of the Cofactor Matrix.
6) Compute Inverse (A^{-1})
7) Check the Inverse by using $A^{-1}A = I$.

Examples:

1. Find the Inverse of the Matrix $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$.

Solution:

Step 1 : Find the determinant of the given Matrix.

Therefore, $|A| = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = 8-3 = 5.$

Since $|A| \neq 0$, the Matrix A is invertible. That is if |A| = 0, we cannot find inverse.

Step 2 : Compute Minors.

$M_{11} = 4 \quad M_{12} = 3 \quad M_{21} = 1 \quad M_{22} = 2.$

Therefore, Minor i.e., $M_{ij} = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$

Step 3 : Compute Cofactor Matrix.

$A_{11} = 4 \quad A_{12} = -3 \quad A_{21} = -1 \quad A_{22} = 2.$

Therefore, Cofactor Matrix i.e., $A_{ij} = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$

Step 4 : Compute Adjoint Matrix i.e., Transpose of the Cofactor Matrix.

Therefore, Adj $A = \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$

Step 5 : Compute Inverse.

$A^{-1} = \frac{\text{Adj } A}{|A|}$ i.e., divide each element in Adj A by the value of the determinant.

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{1}{5} \\ \frac{-3}{5} & \frac{2}{5} \end{bmatrix}$$

Step 6: Check the inverse by using $A^{-1}A = I$.

$$\begin{bmatrix} \frac{4}{5} & -\frac{1}{5} \\ \frac{-3}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. Find the Inverse of the Matrix $A = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$.

Solution:

Step 1 : Find the Determinant of the given Matrix

$|A| = 10 - 6 = 4 \neq 0.$

Since the $|A| \neq 0$, the Matrix A is invertible.

Step 2 : Find Cofactor Matrix.

Therefore $A_{ij} = \begin{bmatrix} 5 & -3 \\ -2 & 2 \end{bmatrix}$

Step 3 : Find Adjoint Matrix i.e., Transpose of the Cofactor Matrix.

Adj $A = \begin{bmatrix} 5 & -2 \\ -3 & 2 \end{bmatrix}$

Step 4 : Find the Inverse.

$A^{-1} = \frac{\text{Adj } A}{|A|}$

Therefore $A^{-1} = \frac{1}{4}\begin{bmatrix} 5 & -2 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & -\frac{2}{4} \\ \frac{-3}{4} & \frac{2}{4} \end{bmatrix}$.

3. Find the Inverse of $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 10 & 2 \\ 3 & 9 & 1 \end{bmatrix}$.

Solution:

Step 1 : Find the Determinant of the given Matrix.

$|A| = 4(10-18) - 0(2-6) + 2(18-30)$

$= 4(-8) - 0(-4) + 2(-12)$

$= -32 - 24 = -56 \neq 0.$

Since $|A| \neq 0$, the Matrix A is invertible.

Step 2 : Find Cofactor Matrix.

$$A_{ij} = \begin{bmatrix} -8 & 4 & -12 \\ 18 & -2 & -36 \\ -20 & -4 & 40 \end{bmatrix}$$

Step 3 : Find Adjoint Matrix i.e., Transpose of the Cofactor Matrix.

$$\text{Adj } A = \begin{bmatrix} -8 & 18 & -20 \\ 4 & -2 & -4 \\ -12 & -36 & 40 \end{bmatrix}$$

Step 4 : Find the Inverse.

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{-56}\begin{bmatrix} -8 & 18 & -20 \\ 4 & -2 & -4 \\ -12 & -36 & 40 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-8}{-56} & \frac{18}{-56} & \frac{-20}{-56} \\ \frac{4}{-56} & \frac{-2}{-56} & \frac{-4}{-56} \\ \frac{-12}{-56} & \frac{-36}{-56} & \frac{40}{-56} \end{bmatrix} = \begin{bmatrix} \frac{8}{56} & -\frac{18}{56} & \frac{20}{56} \\ \frac{-4}{56} & \frac{2}{56} & \frac{4}{56} \\ \frac{12}{56} & \frac{36}{56} & \frac{-40}{56} \end{bmatrix}.$$

4. Find the Inverse of the Matrix $A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & -2 & -3 \\ 3 & -1 & -1 \end{bmatrix}$.

Solution:

Step 1: Find the Determinant for the given Matrix.

$|A| = 0(-2+3) + 1(1+9) + 2(1+6)$

$= 0(1) + 1(10) + 2(7)$

$= 10 + 14 = 24 \neq 0.$

Since $|A| \neq 0$, the Matrix A is invertible.

Step 2 : Find the Cofactor Matrix.

$$A_{ij} = \begin{bmatrix} 1 & -10 & 7 \\ 3 & -6 & -3 \\ 7 & 2 & 1 \end{bmatrix}$$

Step 3 : Find the Adjoint Matrix. i.e., Transpose of the Cofactor Matrix.

$$\text{Adj } A = \begin{bmatrix} 1 & 3 & 7 \\ -10 & -6 & 2 \\ 7 & -3 & 1 \end{bmatrix}$$

Step 4 : Find the Inverse.

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

$$A^{-1} = \frac{1}{24}\begin{bmatrix} 1 & 3 & 7 \\ -10 & -6 & 2 \\ 7 & -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{24} & \frac{3}{24} & \frac{7}{24} \\ \frac{-10}{24} & \frac{-6}{24} & \frac{2}{24} \\ \frac{7}{24} & \frac{-3}{24} & \frac{1}{24} \end{bmatrix}.$$

5. Find the Inverse of the Matrix $A = \begin{bmatrix} 1 & 3 & -4 \\ -1 & -2 & 1 \\ 2 & 4 & -5 \end{bmatrix}$.

Solution:

Step 1 : Find the Determinant for the given Matrix.

$|A| = 1(10-4) - 3(5-2) - 4(-4+4)$

$= 1(6) - 3(3) - 4(0)$

$= 6 - 9 = -3 \neq 0.$

Since, $|A| \neq 0$, the Matrix A is invertible.

Step 2 : Find the Cofactor Matrix.

$$A_{ij} = \begin{bmatrix} 6 & -3 & 0 \\ -1 & 3 & 2 \\ -5 & 3 & 1 \end{bmatrix}$$

Step 3: Find the Adjoint Matrix, i.e., Transpose of the Cofactor Matrix.

$$\text{Adj } A = \begin{bmatrix} 6 & -1 & -5 \\ -3 & 3 & 3 \\ 0 & 2 & 1 \end{bmatrix}$$

Step 4: Find the Inverse

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{|A|}[\text{illegible}]$$

$$A^{-1} = [\text{illegible}] = [\text{illegible}]$$

ii) Find the Inverse of the Matrix

$A = $ [illegible]

Solution

Step 1: Find the Determinant for the given Matrix.

$|A| = 1(-21 + 4) - 2(-15 + 8) + 3(-5 + 14)$

$= 1(-17) - 2(-7) + 3(9)$

$= -17 + 14 + 27 = 24 \neq 0$

Since $|A| \neq 0$ the Matrix A is Invertible.

Step 2: Find the Cofactor Matrix.

$$A_c = \begin{bmatrix} -17 & 7 & 9 \\ -3 & -3 & 3 \\ 13 & -11 & 3 \end{bmatrix}$$

Step 3: Find the Adjoint Matrix, i.e., Transpose of the Cofactor Matrix.

$$\text{Adj } A = \begin{bmatrix} -17 & -3 & 13 \\ 7 & -3 & -11 \\ 9 & 3 & 3 \end{bmatrix}$$

Step 4: Find Inverse. $A^{-1} = \frac{\text{Adj } A}{|A|}$

$$A^{-1} = \frac{1}{24}\begin{bmatrix} -17 & -3 & 13 \\ 7 & -3 & -11 \\ 9 & 3 & 3 \end{bmatrix} = \begin{bmatrix} \frac{-17}{24} & \frac{-3}{24} & \frac{13}{24} \\ \frac{7}{24} & \frac{-3}{24} & \frac{-11}{24} \\ \frac{9}{24} & \frac{3}{24} & \frac{3}{24} \end{bmatrix}$$

Properties of Inverse Matrix

1) $(AB)^{-1} = B^{-1}A^{-1}$

2) $(A^{-1})(A^{-1})^{-1} = I$ i.e., $(A^{-1})A = I$ (since $(A^{-1})^{-1} = A$)

3) $(A^T)^{-1} = (A^{-1})^T$

4) $|A^{-1}| = \frac{1}{|A|}$

EXERCISE 9.1

1. Find 3A − 2B, where A and B are Matrices as $A = [\text{illegible}]$ and $B = [\text{illegible}]$

2. Find 2A − 3B, where A and B are Matrices as $A = \begin{bmatrix} 5 & 2 \\ 7 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & -2 \\ -3 & 5 \end{bmatrix}$

3. If $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} -11 & 0 \\ 13 & -12 \end{bmatrix}$ and $C = \begin{bmatrix} 8 & -9 \\ -17 & 10 \end{bmatrix}$ find 4A−5B+7C.

4. If A is the Matrix $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, evaluate $A^2 - 4A$.

5. Find A^2 and A^3, where $A = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$.

6. Find $(AB)^T$ for the Matrices $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$.

7. Verify $(AB)^T = B^T A^T$, where $A = \begin{bmatrix} 2 & 1 & -4 \\ 0 & 3 & 1 \\ -1 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -3 & 2 \\ 8 & 7 & 0 \\ -2 & 1 & -6 \end{bmatrix}$.

8. Find the inverse of the Matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 3 & -1 & 4 \\ 2 & 1 & -2 \end{bmatrix}$

9. Obtain the Inverse of the Matrix $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 0 & -3 \\ 3 & 4 & 0 \end{bmatrix}$

10. Find the Inverse of the following Matrix $A = \begin{bmatrix} 2 & 4 & 2 \\ 6 & 2 & 1 \\ 1 & 4 & 3 \end{bmatrix}$

11. Determine the Inverse of the Matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 2 \\ 2 & 4 & 2 \end{bmatrix}$

SOLVING A SYSTEM OF LINEAR EQUATIONS

A) MATRIX INVERSION TECHNIQUE

I) Two Simultaneous Equations in Two Unknowns:

Let $A X = B$ be the given equation.

Premultiplying both sides by A^{-1}

$$A^{-1} A X = A^{-1} B$$

$$(A^{-1} A) X = A^{-1} B \text{ ...... (Since Associative Law)}$$

$$I X = A^{-1} B \text{ ...... (Since } A^{-1} A = I)$$

$$X = A^{-1} B$$

Steps to Solve the Linear Equations

1) Write the equations in Matrix form
2) Find the Determinant of the Coefficient Matrix. If $|A| \neq 0$, go to step 3, i.e., if $|A| = 0$, we cannot find inverse.
3) Compute Minors (M_{ij}).
4) Find Cofactor Matrix (A_{ij}).
5) Compute Adjoint Matrix i.e., Transpose of the Cofactor Matrix (Adj A).
6) Find Inverse (A^{-1}).
7) Multiply the Inverse Matrix (A^{-1}) by the Constant Vector (B)i.e., A^{-1} B.

Examples :

1. Solve the following set of Linear Simultaneous Equations.

$$2x_1 + 3x_2 = 5 \text{ .... (1)}$$

$$11x_1 - 5x_2 = 6 \text{ .... (2).}$$

Solution :

Step 1: Write the above equations in Matrix form.

$$\begin{bmatrix} 2 & 3 \\ 11 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

(A) (X) (B)

A – Coefficient Matrix

X – Variable Vector

B – Constant Vector

Step 2: Find the determinant of the Coefficient Matrix.

$$A = \begin{bmatrix} 2 & 3 \\ 11 & -5 \end{bmatrix}$$

$\therefore |A| = -10 - 33 = -43 \neq 0$

Since $A \neq O$, go to step (3).

Step 3: Find Minors

$$M_{11} = -5, \quad M_{12} = 11, \quad M_{21} = 3, \quad M_{22} = 2$$

$$M_{ij} = \begin{bmatrix} -5 & 11 \\ 3 & 2 \end{bmatrix}$$

Step 4: Find Cofactor Matrix.

$$A_{11} = -5, \quad A_{12} = -11, \quad A_{21} = -3, \quad A_{22} = 2$$

$$A_{ij} \begin{bmatrix} -5 & -11 \\ -3 & 2 \end{bmatrix}$$

Step 5: Find Adjoint Matrix (Adj A),
ie., Transpose of the Cofactor Matrix.

$$\begin{bmatrix} -5 & -3 \\ -11 & 2 \end{bmatrix}$$

Step 6: Find Inverse (A^{-1}).

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{-43} \begin{bmatrix} -5 & -3 \\ -11 & 2 \end{bmatrix} = \begin{bmatrix} \frac{-5}{-43} & \frac{-3}{-43} \\ \frac{-11}{-43} & \frac{2}{-43} \end{bmatrix}$$

Step 7: Multiplying the Inverse Matrix (A^{-1}) by the Constant Vector **(B)**

$$\text{i.e., } X = A^{-1} B \begin{bmatrix} \frac{-5}{-43} & \frac{-3}{-43} \\ \frac{-11}{-43} & \frac{2}{-43} \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} \left(\frac{-5}{-43} \times 5\right) + \left(\frac{-3}{-43} \times 6\right) \\ \left(\frac{-11}{-43} \times 5\right) + \left(\frac{2}{-43} \times 6\right) \end{bmatrix}$$

$$\begin{bmatrix} \frac{-25-18}{-43} \\ \frac{-55+12}{-43} \end{bmatrix} = \begin{bmatrix} \frac{-43}{-43} \\ \frac{-43}{-43} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

But $X = A^{-1} B$

i.e., $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Therefore, $x_1 = 1$ and $x_2 = 1$.

Proof :

Substitute the value of x_1 and x_2 in the equation (1)

$$2(1) + 3(1) = 5$$

$$2 + 3 = 5$$

$$5 = 5$$

Hence, answer is correct.

2. Solve for x and y

$$2x + 3y = 7 \text{ .... (1)}$$

$$4x + 2y = 10 \text{ .... (2).}$$

Solution:

Step 1: Write the above equations in Matrix form.

$$\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$$

$$A \quad X = \quad B$$

Step 2: Find the determinant of Coefficient Matrix.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$$

$|A| = 4 - 12 = -8 \neq 0$

Since $|A| \neq 0$, go to step (3)

Step 3: Find Cofactor Matrix

$$\begin{bmatrix} 2 & -4 \\ -3 & 2 \end{bmatrix}$$

Step 4: Find Adjoint Matrix i.e., Transpose of the Cofactor Matrix.

$$\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix}$$

Step 5: Find Inverse (A^{-1}).

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{-8} \begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{-8} & \frac{-3}{-8} \\ \frac{-4}{-8} & \frac{2}{-8} \end{bmatrix} = \begin{bmatrix} \frac{1}{-4} & \frac{-3}{-8} \\ \frac{-1}{-2} & \frac{1}{-4} \end{bmatrix}$$

Step 6: Multiply the Inverse Matrix (A^{-1}) by the Constant Vector (B).

$$X = A^{-1} B = \begin{bmatrix} \frac{1}{-4} & \frac{-3}{-8} \\ \frac{-1}{-2} & \frac{1}{-4} \end{bmatrix} \begin{bmatrix} 7 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1}{-4} \times 7\right) + \left(\frac{-3}{-8} \times 10\right) \\ \left(\frac{-1}{-2} \times 7\right) + \left(\frac{1}{-4} \times 10\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{-4} + \frac{-30}{-8} \\ \frac{-7}{-2} + \frac{10}{-4} \end{bmatrix} = \begin{bmatrix} \frac{14-30}{-8} \\ \frac{-14+10}{-4} \end{bmatrix} = \begin{bmatrix} \frac{-16}{-8} \\ \frac{-4}{-4} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

But $X = A^{-1} B$

ie., $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Answer: $x = 2$ and $y = 1$

Proof :

Substitute the value of x and y in equation (1)

$$2x + 3y = 7$$
$$2(2) + 3(1) = 7$$
$$4 + 3 = 7$$
$$7 = 7$$

Hence, the answer is correct.

II) Three Simultaneous Linear Equations in Three Unknowns :

Examples:

1. Solve :

$2x - 4y + 3z = 3$ .... (1)

$4x - 6y + 5z = 2$ .... (2)

$-2x + y - z = 1$ .... (3).

Solution:

Step 1: Write the above equations in Matrix form.

$$\begin{bmatrix} 2 & -4 & 3 \\ 4 & -6 & 5 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$(A) \quad (X) = (B)$$
$$AX = B$$

Step 2: Find the Determinant of the Coefficient Matrix.

$|A| = 2(6-5) + 4(-4+10) + 3(4-12)$

$= 2(1) + 4(6) + 3(-8) = 2 + 24 - 24 = 2 \neq 0$

Since $|A| \neq 0$, go to step (3).

Step 3: Find the Cofactor Matrix.

$$\begin{bmatrix} 1 & -6 & -8 \\ -1 & 4 & 6 \\ -2 & 2 & 4 \end{bmatrix}$$

Step 4: Find the Adjoint Matrix i.e., Transpose of the Cofactor Matrix.

$$\begin{bmatrix} 1 & -1 & -2 \\ -6 & 4 & 2 \\ -8 & 6 & 4 \end{bmatrix}$$

Step 5: Find Inverse (A^{-1})

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{2} \begin{bmatrix} 1 & -1 & -2 \\ -6 & 4 & 2 \\ -8 & 6 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{-2}{2} \\ \frac{-6}{2} & \frac{4}{2} & \frac{2}{2} \\ \frac{-8}{2} & \frac{6}{2} & \frac{4}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & -1 \\ -3 & 2 & 1 \\ -4 & 3 & 2 \end{bmatrix}$$

Step 6: Multiply the Inverse Matrix (A^{-1}) by the Constant Vector (B).

$$X = A^{-1}B = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & -1 \\ -3 & 2 & 1 \\ -4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} \times 3\right) + \left(\frac{-1}{2} \times 2\right) + (-1 \times 1) \\ (-3 \times 3) + (2 \times 2) + (1 \times 1) \\ (-4 \times 3) + (3 \times 2) + (2 \times 1) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} - \frac{2}{2} - 1 \\ -9 + 4 + 1 \\ -12 + 6 + 2 \end{bmatrix} = \begin{bmatrix} \frac{3-2-2}{2} \\ -9+4+1 \\ -4 \end{bmatrix} = \begin{bmatrix} -0.5 \\ -4 \\ -4 \end{bmatrix}$$

But $X = A^{-1}B$

ie., $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -0.5 \\ -4 \\ -4 \end{bmatrix}$

Answer : $x = -0.5$, $y = -4$ and $z = -4$.

Proof : Substitute the value of x,y,z in the equation (1)

$$2x - 4y + 3z = 3$$
$$2(-0.5) - 4(-4) + 3(-4) = 3$$
$$-1 + 16 - 12 = 3$$
$$16 - 13 = 3$$
$$3 = 3$$

Hence, the answer is correct.

2. Solve the following set of Linear Simultaneous Equations:

$2x_1 + 4x_2 - x_3 = 15$ .... (1)

$x_1 - 3x_2 + 2x_3 = -5$ .... (2)

$6x_1 + 5x_2 + x_3 = 28$ .... (3).

Solution:

Step 1 : Write these equations in Matrix form.

$$\begin{bmatrix} 2 & 4 & -1 \\ 1 & -3 & 2 \\ 6 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ -5 \\ 28 \end{bmatrix}$$

Step 2: Find the determinant of the Coefficient Matrix.

$$A = \begin{bmatrix} 2 & 4 & -1 \\ 1 & -3 & 2 \\ 6 & 5 & 1 \end{bmatrix}$$

$$|A| = 2(-3-10) - 4(1-12) - 1[5 - (-18)]$$
$$= 2(-13) - 4(-11) - 1(5+18) = -26 + 44 - 23 = -5 \neq 0$$

Since $|A| \neq 0$, go to step (3)

Step 3: Find Cofactor Matrix

$$A_{ij} = \begin{bmatrix} -13 & 11 & 23 \\ -9 & 8 & 14 \\ 5 & -5 & -10 \end{bmatrix}$$

Step 4 : Find Adjoint Matrix, i.e., Transpose of the Cofactor Matrix.

$$\text{Adj } A = \begin{bmatrix} -13 & -9 & 5 \\ 11 & 8 & -5 \\ 23 & 14 & -10 \end{bmatrix}$$

Step 5 : Find the Inverse.

$$A^{-1} = \frac{\text{Adj } A}{|A|} = -\frac{1}{5}\begin{bmatrix} -13 & -9 & 5 \\ 11 & 8 & -5 \\ 23 & 14 & -10 \end{bmatrix} = \begin{bmatrix} \frac{-13}{-5} & \frac{-9}{-5} & \frac{5}{-5} \\ \frac{11}{-5} & \frac{8}{-5} & \frac{-5}{-5} \\ \frac{23}{-5} & \frac{14}{-5} & \frac{-10}{-5} \end{bmatrix}$$

Step 6 : Multiply the Inverse Matrix (A^{-1}) by the Constant Vector (B) i.e.,

$$X = A^{-1}B = \begin{bmatrix} \frac{-13}{-5} & \frac{-9}{-5} & \frac{5}{-5} \\ \frac{11}{-5} & \frac{8}{-5} & \frac{-5}{-5} \\ \frac{23}{-5} & \frac{14}{-5} & \frac{-10}{-5} \end{bmatrix} \begin{bmatrix} 15 \\ -5 \\ 28 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{-13}{-5} \times 15\right) + \left(\frac{-9}{-5} \times -5\right) + \left(\frac{5}{-5} \times 28\right) \\ \left(\frac{11}{-5} \times 15\right) + \left(\frac{8}{-5} \times -5\right) + \left(\frac{-5}{-5} \times 28\right) \\ \left(\frac{23}{-5} \times 15\right) + \left(\frac{14}{-5} \times -5\right) + \left(\frac{-10}{-5} \times 28\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-195}{-5} + \frac{45}{-5} + \frac{140}{-5} \\ \frac{165}{-5} + \frac{-40}{-5} + \frac{-140}{-5} \\ \frac{345}{-5} + \frac{-70}{-5} + \frac{-280}{-5} \end{bmatrix} = \begin{bmatrix} \frac{-195 + 45 + 140}{-5} \\ \frac{165 - 40 - 140}{-5} \\ \frac{345 - 70 - 280}{-5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-10}{-5} \\ \frac{-15}{-5} \\ \frac{-5}{-5} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

But $X = A^{-1}B$

i.e., $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

Therefore $x_1 = 2$, $x_2 = 3$ and $x_3 = 1$

Proof :

Substitute the value of x_1, x_2, and x_3 in the equation (1)

$2x_1 + 4x_2 - x_3 = 15$

$2(2) + 4(3) - 1(1) = 15$

$4 + 12 - 1 = 15$

$15 = 15$

Hence, the answer is correct.

3. Solve the following set of **Linear Simultaneous Equations:**

$2x - 3y + 4z = 5$ .... (1)

$x + 2y - 3z = 8$ .... (2)

$x - y - z = 1$ .... (3).

Solution :

Step 1: Write the above equations in **Matrix form.**

$$\begin{bmatrix} 2 & -3 & 4 \\ 1 & 2 & -3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 1 \end{bmatrix}$$

Step 2: Find the determinant of **the Coefficient Matrix.**

$$A = \begin{bmatrix} 2 & -3 & 4 \\ 1 & 2 & -3 \\ 1 & -1 & -1 \end{bmatrix}$$

$|A| = 2(-2-3) + 3(-1+3) + 4(-1-2)$

$= -10 + 6 - 12 = -22 + 6 = -16 \neq 0$

Since $|A| \neq 0$, go to step (3)

Step 3: Find the Cofactor Matrix.

$$A_{ij} = \begin{bmatrix} -5 & -2 & -3 \\ -7 & -6 & -1 \\ 1 & 10 & 7 \end{bmatrix}$$

Step 4: Find the Adjoint Matrix i.e., Transpose of the Coefficient Matrix

$$\text{Adj } A = \begin{bmatrix} -5 & -7 & 1 \\ -2 & -6 & 10 \\ -3 & -1 & 7 \end{bmatrix}$$

Step 5: Find the Inverse.

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{-16} \begin{bmatrix} -5 & -7 & 1 \\ -2 & -6 & 10 \\ -3 & -1 & 7 \end{bmatrix} = \begin{bmatrix} \frac{-5}{-16} & \frac{-7}{-16} & \frac{1}{-16} \\ \frac{-2}{-16} & \frac{-6}{-16} & \frac{10}{-16} \\ \frac{-3}{-16} & \frac{-1}{-16} & \frac{7}{-16} \end{bmatrix}$$

Step 6: Multiply the Inverse Matrix (A^{-1}) by the Constant Vector (B).

$$X = A^{-1}B = \begin{bmatrix} \frac{5}{16} & \frac{7}{16} & \frac{1}{-16} \\ \frac{2}{16} & \frac{6}{16} & \frac{10}{-16} \\ \frac{3}{16} & \frac{1}{16} & \frac{7}{-16} \end{bmatrix} \begin{bmatrix} 5 \\ 8 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{16} \times 5 + \frac{7}{16} \times 8 + \left(\frac{1}{-16} \times 1\right) \\ \frac{2}{16} \times 5 + \frac{6}{16} \times 8 + \left(\frac{10}{-16} \times 1\right) \\ \frac{3}{16} \times 5 + \frac{1}{16} \times 8 + \left(\frac{7}{-16} \times 1\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{25}{16} + \frac{56}{16} + \left(\frac{1}{-16}\right) \\ \frac{10}{16} + \frac{48}{16} + \left(\frac{10}{-16}\right) \\ \frac{15}{16} + \frac{8}{16} + \left(\frac{7}{-16}\right) \end{bmatrix} = \begin{bmatrix} \frac{25 + 56 - 1}{16} \\ \frac{10 + 48 - 10}{16} \\ \frac{15 + 8 - 7}{16} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{80}{16} \\ \frac{48}{16} \\ \frac{16}{16} \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

But $X = A^{-1}B$ i.e., $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$

Therefore, $x = 5$, $y = 3$, and $z = 1$.