Replacement

In real life we come across a number of situations wherein items are to be replaced because of their inefficiencies to serve economically or because of sudden failure or break downs. Hence there is a need to determine a most economical replacement policy which will help to serve most efficiently. Generally we come across such replacements for machines, trucks, electric bulbs, fuses etc., and in problems of mortality and staffing.

There are two types of failures that are considered in replacement theory They are

(i) Gradual failure

(ii) Sudden failure

Gradual Failure: The failure mechanism under this category is progressive, that is as the life of an item increases, its efficiency decreases resulting in

(i) increased expenditure for operating costs

(ii) decreased output of the machine

(iii) decrease in resale value of the machine

Such situation arise in items like cars, machines, trucks, automobile tyres, etc.

Sudden Failure: This class of failure is not applicable to items that deteriorates with time but to items that fail suddenly after a period of use. But this period is not a constant but may follow a probability distribution which may be progressive, retrogressive or random.

Progressive failure: Under this category, the probability of failure increases with the age of the item. For example such a thing occurs in electrical bulbs, automobile tyres, etc.

Retrogressive failure: There are items having greater probability of failure in the early life and the chance of failure decreases as age increases. In otherwords the ability of the item to survive the initial period of life increases its expected life. This types of failure occurs in aircraft engines.

Random failure: There are items where in failure may occur randomly independent of the age of the item. For example vacum tubes in air-borne equipment have been shown to fail at a rate independent of the age of the tube.

Preventive Replacement: There are items which may form a part of large complete system. If any of the item fails, then the entire system breaks down. The sudden failure of such items may result in loss of production, idle inventory, labour and other such losses. If it is possible to predict as to when such item is going to fail then replacing such item before its failure is possible prevailing the sudden breakdown of the system. Such a case is known as preventive replacement.

Strategies of replacement: There are three types of replacement policies which follow sudden failure mechanism.

They are (i) Replacement of an item as when it fails.

(ii) Individual preventive replacement

(iii) Common preventive replacement.

In the first situation we come across items which breakdown at different times. As and when they breakdown a new item is replaced or the old one is repaired. This may arise in vacuum tubes, transistors, etc.

Individual preventive replacement

According to this an item has to be replaced immediately after its failure or preventive maintenance is performed in it by knowing its optimum life. That is the item is replaced after its known optimum life period even though it may still survive.

Common preventive replacement: This strategy is applied for a system consisting of a group of items. Here the individually failed items are replaced as and when they fail and all the items are replaced after the optimal period. This situation is also called *group replacement*.

Next our interest will be to determine the age of replacement of such items. In case of items which deteriorates with time we consider the following two situations.

(i) Replacement of items whose maintenance cost increases with time and money value is not considered.

(ii) Replacement of items whose maintenance cost increases with time and money value changes with time.

We shall now derive formula for replacement of items in the above two cases.

For a machine, the maintenance cost always increases with time and a stage comes when the maintenance cost is so large. It is uneconomical to continue the item in service and hence we have to replace this item by a new one. Hence we have to determine the best age at which the replacement should take place.

Model I

Replacement of an item whose maintenance cost increases with time and money value is not changed.

Let C be the cost of the item

$f(t)$ be the operating cost during the period t.

$s(t)$ be the scrap value (resale value) after the period t.

which the average is minimum. ... the year at

Example 1: Following table gives the running costs per year and resale price of a certain equipment whose purchase price Rs. 5,000.

Year	1	2	3	4	5	6	6	7
Running cost (Rs.)	1500	1600	1800	2100	2500	2900	3400	4000
Resale value (Rs.)	3500	2500	1700	1200	800	500	500	500

At what year is the replacement due?

(CA May 1985)

Solution:

Year	$f(t)$	$\Sigma f(t)$	$s(t)$	$c-s(t)$	T	T_A
1	1500	1500	3500	1500	3000	3000
2	1600	3100	2500	2500	5600	2800
3	1800	4900	1700	3300	8200	2733
4	2100	7000	1200	3800	10800	2700
5	2500	9500	800	4200	13700	2740
6	2900	12,400	500	4500	16900	2817
7	3400	15,800	500	4500	20300	2900
8	4000	19,800	500	4500	24300	3038

The average & cost reaches its minimum value at the end of the 4th year. Hence the equipment should be replaced at the end of every 4th year.

Example 2: The following table gives the costs per year and the resale value of a certain equipment whose purchase price is Rs. 6500. At what age is the replacement due optimally.

Year	1	2	3	4	5	6	7	8
Running costs (Rs.)	1400	1500	1700	2000	2400	2800	3300	3900
Resale value (Rs.)	4000	3000	2200	1700	1300	1000	1000	1000

Solution:

Year	$f(t)$	$\Sigma f(t)$	$s(t)$	$c-s(t)$	T	T_A
1	1400	1400	4000	2500	3900	3900
2	1500	2900	3000	3500	6400	3200
3	1700	4600	2200	4300	8900	2967
4	2000	6600	1700	4800	11400	2850
5	2400	9000	1300	5200	14200	2840
6	2800	11800	1000	5500	17300	2883
7	3300	15100	1000	3500	20600	2943
8	3900	19000	1000	5500	24500	3063

The average cost reaches its minimum value at the end of the 5th year. Hence the equipment should be replaced at the end of 5th year, *i.e.*, when its age is 5.

Example 3: The cost of a machine is Rs. 6100 and its scrap value is Rs. 100. The maintenance costs found from experience are as follows

Year	1	2	3	4	5	6	7	8
Maintenance costs (Rs.)	100	250	400	600	900	1200	1600	2000

When should the machine be replaced?

Solution:

Year	$f(t)$	$\Sigma f(t)$	$s(t)$	$c-s(t)$	T	T_A
1	100	100	100	6000	6100	6100
2	250	350	100	6000	6350	3175
3	400	750	100	6000	6750	2250
4	600	1350	100	6000	7350	1837
5	900	2250	100	6000	8250	1650
6	1200	3450	100	6000	9450	1575
7	1600	5050	100	6000	11050	1579
8	2000	7050	100	6000	13050	1631

The average cost reaches its minimum value at the end of the 6th year. The machine should be replaced at the end of every 6th year.

Example 4: A truck owner finds from his past records that the maintenance costs per year of a truck whose purchase price is Rs. 8000 are as given below:

Year	1	2	3	4	5	6	7	8
Maintenance cost cost (Rs.)	1000	1300	1700	2200	2900	3800	4800	6000
Resale value (Rs.)	4000	2000	1200	600	500	400	400	400

Determine the time it is profitable to replace the truck

Solution:

Year	$f(t)$	$\Sigma f(t)$	$s(t)$	$c-s(t)$	T	T_A
1	1000	1000	4000	4000	5000	5000
2	1300	2300	2000	6000	8300	4150
3	1700	4000	1200	6800	12800	4267
4	2200	6200	600	7400	13600	3400
5	2900	9100	500	7500	16600	3320
6	3800	12900	400	7600	20500	3417
7	4800	17700	400	7600	25300	3614
8	6000	23700	400	7600	31300	3915

The average cost reaches its minimum value at the end of the 5th year. Hence the time of profitable replacement is the end of 5th year.

Example 5: For a machine, the following data are available.

Year	0	1	2	3	4	5	6
Cost of spares (Rs.)	–	200	400	700	1000	1400	1600
Salary of maintenance staff (Rs.)	–	1200	1200	1400	1600	2000	2600
Losses due to break down (Rs.)	–	600	800	700	1000	1200	1600
Resale value (Rs.)	12000	6000	3000	1500	800	400	400

Determine the optimum period for replacement of the above machine.

(CA Nov. 1987)

Solution: Here f(t)

= Cost of sparés + Salary of maintenance staff + losses due to breakdown

Year	$f(t)$	$\Sigma f(t)$	$s(t)$	$c-s(t)$	T	T_A
1	2000	2000	6000	6000	8000	8000
2	2400	4400	3000	9000	13400	6700
3	2800	7200	1500	10500	17700	5900
4	3600	10800	800	11200	22000	5500
5	4600	15400	400	11600	27000	5400
6	5800	21200	400	11600	32800	5467

The average cost reaches its minimum value at the end of the 5th year.

∴ The optimum replacement of the machine is the end of the 5th year.

Example 6: Following table gives the running costs per year and resale price of a vehicle whose purchase price is Rs. 5200.

Year	1	2	3	4	5	6	7
Resale value (Rs.)	3500	2700	1800	1000	850	600	425
Running costs (Rs.)	600	850	1000	1250	1400	1475	2000

At what year is the replacement due?

(ICWA Dec. 1985)

Solution :

Year	$f(t)$	$\Sigma f(t)$	$s(t)$	$c-s(t)$	T	T_A
1	600	600	3500	1700	2300	2300
2	850	1450	2700	2500	3950	1975
3	1000	2450	1800	3400	5850	1950
4	1250	3700	1000	4200	7900	1975
5	1400	5100	850	4350	9450	1890
6	1475	6575	600	4600	11,175	1863
7	2000	8575	425	4775	13,350	1907

The replacement is due at the end of 6 years

Example 8: An equipment which costs Rs. 15000 has to be replaced with a new equipment. The following data has been estimated

Year	1	2	3	4	5	6	7	8
Resale value	12000	9500	7500	5700	4200	3900	2900	2000
Annual maintenance cost (Rs.)	600	800	1050	1400	2100	3500	5000	6500

Ignore the time value of money and inflation.

Solution :

Year	$f(t)$	$\Sigma f(t)$	$s(t)$	$c-s(t)$	T	T_A
1	600	600	12000	3000	3600	3600
2	800	1400	9500	5500	6900	3450
3	1050	2450	7500	7500	9950	3317
4	1400	3850	5700	9300	13150	3288
5	2100	5950	4200	10800	16750	3350
6	3500	9450	3900	11100	20550	3425
7	5000	14450	2900	12100	26550	3793
8	6800	23250	2000	13000	36250	4531

The average cost reaches its minimum value at the end of the 4th year. Hence the equipment has to be replaced at the end of the 4th year.

Example 9: An electro-mechanical equipment has a purchase price of Rs. 7000. The running costs per year and resale values are given below:

Year	1	2	3	4	5	6	7	8
Running cost (Rs.)	2000	2100	2300	2600	3000	3500	4100	4600
Resale value (Rs.)	4000	3000	2200	1600	1400	700	700	700

At which year is the replacement due?

(ICWA, Dec. 1986)

Solution :

Year	$f(t)$	$\Sigma f(t)$	$s(t)$	$c-s(t)$	T	T_A
1	2000	2000	4000	3000	5000	5000
2	2100	4100	3000	4000	8100	4050
3	2300	6400	2200	4800	11200	3733
4	2600	9000	1600	5400	14400	3600
5	3000	12000	1400	5600	17600	3520
6	3500	15500	700	6300	21800	3633
7	4100	19600	700	6300	25900	3700
8	4600	24200	700	6300	30500	3813

The average cost reaches its minimum value at the end of 5th year. Hence the equipment has to be replaced at the end of 5th year.

Example 10: The data collected in running a machine the cost of which is Rs 60,000 are given below:

Year	1	2	3	4	5
Real value (Rs.)	42,000	20,000	18400	14400	9650
Cost of spares (Rs.)	4000	4270	4880	5700	6800
Cost of labour (Rs.)	14000	16000	18000	21000	25000

Determine optimum period for replacement of the machine.

(ICWA, June 1990)

Solution:

Year	$f(t)$	$\Sigma f(t)$	$c-s(t)$	T	T_A
1	18000	18000	42000	60,000	60,000
2	20270	38270	20,000	58270	29135
3	22880	61150	18400	79550	26517
4	26700	87850	14400	102250	25563
5	31800	49600	9650	129250	25850

The average cost is minimum at the end of the 4th year. Hence the machine should be replaced at the end of the 4th year.

Example 11: (a) Machine A costs Rs. 9000. Annual operating costs are Rs. 200 for the first year and then increase by Rs. 2000 every year. Determine the best age at which to replace the machine. If the optimum replacement policy is followed, what will be the average yearly cost of owning and operating the machine?

(b) Machine B costs Rs. 10,000. Annual operating costs are Rs. 400 for the first year and then increases by Rs. 800 every year. You now have a machine of type A which is one year old. Should you replace it with B. If so when?

Solution: (a) For machine A,

Year	$f(t)$	$\Sigma f(t)$	$s(t)$	$c-s(t)$	T	T_A
1	200	200	—	9000	9200	9200
2	2200	2400	—	9000	11400	5700
3	4200	6600	—	9000	15600	5200
4	6200	12800	—	9000	21800	5450
5	8200	21000	—	9000	30000	6000
6	10200	31200	—	9000	40200	10050

∴ The average cost reaches its minimum value at the end of the 3rd year.

∴ The machine A should be replaced at the end of every 3rd year.

(b) For machine B

Year	$f(t)$	$\Sigma f(t)$	$s(t)$	$c-s(t)$	T	T_A
1	400	400	–	10000	10400	10400
2	1200	1600	–	10000	11600	5800
3	2000	3600	–	10000	13600	4533
4	2800	6400	–	10000	16400	4100
5	3600	10,000	–	10000	20,000	4000
6	4400	14400	–	10000	24,400	4067

The average cost reaches its minimum value at the end of the 5th year. Hence machine B should be replaced at the end of 5th year.

Since the average yearly cost of machine B (Rs. 4000) is less than the average yearly cost of machine A (Rs. 5200), machine B should be replaced by machine A. To find the time of replacement we have to compare the average cost of machine B with the yearly cost of A. For a one year old machine A, the yearly cost in the 2nd year is Rs. 2200, which is less than the average cost of machine B (Rs. 4000). Therefore continue machine A for the 2nd year.

The average cost of machine A for the 3rd year is Rs. 5200 which is more than the average cost of machine B (Rs. 4000). $\therefore$ Replace machine A by machine B at the end of the 2nd year; that is when its age is two.

Example 12: The data on the operating costs per year and resale prices of equipment A whose purchase price is Rs. 10000 are given below:

Year	1	2	3	4	5	6	7
Operating cost (Rs.)	1500	1900	2300	2900	3600	4500	5500
Resale value (Rs.)	5000	2500	1250	600	400	400	400

(i) What is the optimum period for replacement?

(ii) When equipment A is 2 years old equipment B, which is a new model for the same usage is available. The optimum period for replacement is 4 years with an average cost Rs. 3600 should we change equipment A with that of B? If so when?

(CA May 1986)

Solution :

Year	$f(t)$	$\Sigma f(t)$	$s(t)$	$c-s(t)$	T	T_A
1	1500	1500	5000	5000	6500	6500
2	1900	3400	2500	7500	10900	5450
3	2300	5700	1250	8750	14450	4817
4	2900	8600	600	9400	18000	4500
5	3600	12200	400	9600	21800	4360
6	4500	16700	400	9600	26300	4383
7	5500	22200	400	9600	31800	4543

Model 3 : Replacement of items due to sudden failure 9

Earlier we considered the models for replacement of an item which deteriorates with time and because of this the maintenance cost of the item increases with time. Now we consider the replacement of a item because of its sudden failure. I he sudden failure of an item (or items) may result in the complete breakdown of the system and hence there arises the need to determine a replacement policy. In such cases two types of policies are followed. They are

(i) Individual replacement policy

(ii) Group replacement policy

According to individual policy the failed item is replaced immediately by a new item. For example in our houses we replace a fused bulb immediately by a new bulb.

Group replacement policy: This policy is followed when we use a large number of items at a time in a system.

Example : Number of light bulbs in a factory, number of transistors in a television or computer, etc. In such a situation the question of interest is to know when all the items should be replaced. We may know the probability distribution of life time of the items in the system. This may help us to determine the period of group replacement. The advantage of group replacement is that the cost of a item when the items are purchased in bulk will be much less than the cost of an individuals item.

Let us now derive the optimal replacement period in the case of group replacement.

Let N be the total number of items in the system.

Let N_i = Number of items failed during the ith period, $i = 1, 2, 3, \ldots\ldots (n-1)$

Let C_1 = The cost of replacing an individual item.

C_g = The total cost per item in the case of group replacement.

$C(n)$ = Total cost of replacing the items (group replacement cost + individual replacement cost) in an interval consisting of n periods.

Then $C(n) = NC_g + C_1[N_i + N_2 + \ldots\ldots + N_{n-1}]$

$$= NC_g + C_1 \sum_{i=1}^{n-1} N_i \qquad \ldots\ldots\ldots (1)$$

From (5) we conclude that

(1) Group replacement should be made at the end of period n if the cost of individual replacement for the nth period is greater than the average cost per period at the end of n periods.

(2) Group replacement should not be made at the end of nth period if the cost of individual replacement at the end of $(n-1)$th period is not less than the average cost per period at the end of $(n-1)$ periods.

Example 1: Following mortality rates have been observed for a certain type of fuses.

Week	1	2	3	4	5
% failing before end of week	5	15	35	75	100

There are 1000 fuses in use and it costs Rs. 5 to replace an individual fuse. If all fuses were replaced simultaneously it would cost Rs. 1.25 per fuse. It is proposed to replace all fuses at fixed intervals of time, whether or not they have burnt out, and to continue replacing burnt out fuses as they fail. At what intervals the group replacement should be made? Also prove that this optimal policy is superior to the straight forward policy of replacing each fuse only when it fails.

(C.A. May 1989)

Solution:

Week	*% failing before the end of week*	*% failing during the week*	*Prob. of failing during the week*
1	5	5	0.05
2	15	10	0.10
3	35	20	0.20
4	75	40	0.40
5	100	25	0.25

Let P_i be the probability that a fuse fails during the ith week.

Then $P_1 = 0.05$; $P_2 = 0.10$; $P_3 = 0.20$; $P_4 = 0.40$ and $P_5 = 0.25$.

Let N_i be the number of fuses failed during the ith week.

Then
$$N_0 = 1000$$
$$N_1 = N_0 P_1 = 1000 \times 0.05 = 50$$
$$N_2 = N_0 P_2 + N_1 P_1$$
$$= 1000 \times 0.10 + 50 \times 0.05 = 100 + 2.5 \approx 102$$
$$N_3 = N_0 P_3 + N_1 P_2 + N_2 P_1$$
$$= 1000 \times 0.20 + 50 \times 0.10 + 102 \times 0.05$$
$$= 200 + 5 + 5 = 210$$

$$N_4 = N_0P_4 + N_1P_3 + N_2P_2 + N_3P_1$$
$$= 1000 \times 0.4 + 50 \times 0.20 + 102 \times 0.10 + 210 \times 0.05$$
$$= 400 + 10 + 10 + 10.5 = 430$$
$$N_5 = N_0P_5 + N_1P_4 + N_2P_3 + N_3P_2 + N_4P_1$$
$$= 1000 \times 0.25 + 50 \times 0.40 + 102 \times 0.2 + 210 \times 0.10 + 430 \times 0.05$$
$$= 250 + 20 + 20 + 21 + 22 = 333$$

When all the fuses are replaced the cost is Rs. 1.25 per fuse. Let us calculate the total cost of group replacement.

End of week	*Total cost*	*Average cost per week*
1	$1000 \times 1.25 + 50 \times 5$ $= 1250 + 250 = 15000$	1550
2	$1500 + 102 \times 5$ $= 1500 + 510 = 2010$	1005
3	$2010 + 210 \times 5$ $= 2010 + 1050 = 3060$	1020
4	$3060 + 430 \times 5$ $= 3050 + 2150 = 5210$	1302.50
5	$5210 + 333 \times 5$ $= 5210 + 1665 = 6875$	1375

The average cost has reached its minimum value at the end of 2 weeks.

∴ Group replacement period is 2 weeks. The minimum average cost per week is Rs. 1005.

In order to show that this optimal policy is superior to individual replacement policy we have to calculate the average cost of individual replacement. First let us find the average life time of a fuse.

Week	*Probability of failure*	*Expected life*
1	0.05	0.05
2	0.10	0.20
3	0.20	0.60
4	0.40	1.60
5	0.25	1.25
		3.70

∴ Average number of fuses failing per week $= \dfrac{1000}{3.7} = 270$

∴ Total average cost of individual replacement $= 270 \times 5 =$ Rs. 1350

Comparing this with the average cost of group replacement we find that the group replacement policy is superior to individual replacement policy.

Example 2: The following failure rates have been observed for a certain type of transistors in a digital computer.

End of week	1	2	3	4	5	6	7	8
Probability of failure to date	0.05	0.13	0.25	0.43	0.68	0.88	0.96	1.0

The cost of replacing an individual failed transistor is Rs. 1.25. The decision is made to replace all these transistors simultaneously at fixed intervals and to replace the individual tansistors as they fail in service. If the cost of group replacement is 30 paise per transistor what is the interval between group replacement? Is it preferable over individual replacement policy?

(C.A. May 1981)

Solution:

Let P_i be the probability that a transistor may fail during the ith week. Then the probability distribution of failures is given below:

Week	*Cumulative probability*	*Probability* P_i
1	0.05	0.05
2	0.13	0.08
3	0.25	0.12
4	0.43	0.18
5	0.68	0.25
6	0.88	0.20
7	0.96	0.08
8	1.00	0.04

Assume that 1000 transistors are in use. Let N_i be the number of failure during the ith week.

$N_0 = 1000$

$N_1 = N_0 P_1 = 1000 \times 0.05 = 50$

$N_2 = N_0 P_2 + N_1 P_1$

$= 1000 \times 0.08 + 50 \times 0.05 = 82$

$N_3 = N_0 P_3 + N_1 P_2 + N_2 P_1$

$= 1000 \times 0.12 + 50 \times 0.08 + 82 \times 0.05 = 128$

$N_4 = N_0 P_4 + N_1 P_3 + N_2 P_2 + N_3 P_1$

$= 1000 \times 0.18 + 50 \times 0.12 + 82 \times 0.08 + 128 \times 0.05 = 199$

$N_5 = N_0 P_5 + N_1 P_4 + N_2 P_3 + N_3 P_2 + N_4 P_1$

$= 1000 \times 0.25 + 50 \times 0.18 + 82 \times 0.12 + 128 \times 0.08 + 199 \times 0.05$

$= 281$

Similarly we can calculate N_6, N_7, N_8

$N_6 = 272$

$N_7 = 194$

$N_8 = 195$

Cost of group replacement

Week	*Total cost*	*Average cost*
1	$1000 \times 0.30 + 50 \times 1.25$ = *363 nearly*	*363*
2	$363 + 82 \times 1.25 = 465$	*232.50*
3	$465 + 128 \times 1.25 = 625$	*208.30*
4	$625 + 197 \times 1.25 = 874$	*218.50*

The average cost per week reaches its minimum value at the end of 3 weeks

∴ The period of group replacement is 2 weeks and the minimum cost per week is Rs. 208.30.

Let us compare it with individual replacement policy.

First let us calculate the average life time of a transistor.

Week	*Probability*	*Average life time*
1	0.05	0.05
2	0.08	0.16
3	0.12	0.36
4	0.18	0.72
5	0.25	1.25
6	0.20	1.20
7	0.08	0.56
8	0.04	0.32
		4.62

Average life time of a transistor = 4.62 weeks

∴ Average number of failures per week $= \dfrac{1000}{4.61} = 216$

∴ Average cost of individual replacement = Rs. 216 × 1.25 = Rs. 270

Comparing the average cost of individual replacement with that of group replacement we find that group replacement is preferable.